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# Characterizations of compact sets in fuzzy sets spaces with $L_p$ metric ☆

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**Abstract**

Compactness criteria for fuzzy sets spaces endowed with $L_p$ metric have been studied for several decades. In metric spaces, totally boundedness is a key feature of compactness. However, compare the existing compactness criteria for fuzzy sets spaces endowed with $L_p$ metric with Arzelà–Ascoli theorem, we can see that the latter gives the compactness criteria by characterizing the totally bounded sets while the former does not seem to characterize the totally bounded sets. Besides, till now, compactness criteria are only presented for three particular fuzzy sets spaces, they both have assumptions of convexity or star-shapedness. In recent years, general fuzzy sets become more and more important in both theory and applications. Motivated by needs listed above, in this paper, we present characterizations of totally bounded sets, relatively compact sets and compact sets in the fuzzy sets spaces $F_B(\mathbb{R}^m)$ and $F_B(\mathbb{R}^m)^p$ equipped with $L_p$ metric, where $F_B(\mathbb{R}^m)$ and $F_B(\mathbb{R}^m)^p$ are two kinds of general fuzzy sets on $\mathbb{R}^m$ which do not have any assumptions of convexity or star-shapedness. Subsets of $F_B(\mathbb{R}^m)^p$ include common fuzzy sets such as fuzzy numbers, fuzzy star-shaped numbers with respect to the origin, fuzzy star-shaped numbers, and the general fuzzy star-shaped numbers introduced by Qiu et al. The existed compactness criteria are stated for three kinds of fuzzy sets spaces endowed with $L_p$ metric whose universe sets are the former three kinds of common fuzzy sets respectively. Construct-

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ing completions of fuzzy sets spaces with respect to $L_p$ metric is a problem which is closely dependent on characterizing totally bounded sets. Based on preceding characterizations of totally boundedness and relatively compactness and some discussions on convexity and star-shapedness of fuzzy sets, we show that the completions of fuzzy sets spaces mentioned in this paper can be obtained by using the $L_p$-extension. We also clarify relation among all the ten fuzzy sets spaces discussed in this paper, which consist of five pairs of original spaces and the corresponding completions. Then, we show that the subspaces of $F_B(\mathbb{R}^m)$ and $F_B(\mathbb{R}^m)^p$ mentioned in this paper have parallel characterizations of totally bounded sets, relatively compact sets and compact sets. At last, as applications of our results, we discuss properties of $L_p$ metric on fuzzy sets space and relook compactness criteria proposed in previous work.

*Keywords:* Fuzzy sets; Compact sets; Totally bounded sets; $L_p$ metric; Completions

## 1. Introduction

Compactness is a fundamental property in both theory and applications [15, 22, 30]. The research of compactness criteria attracts much attention. It's well-known that Arzelà–Ascoli theorem(s) provide compactness criteria in classic analysis and topology. Fuzzy systems have been successfully used to solve many real-world problems [1, 5, 6, 14, 36]. Undoubtedly, compactness plays an important role in analysis and applications of fuzzy sets and systems [4, 17, 18, 20, 29, 33, 36]. There exist many important and interesting works including [7, 8, 10, 12, 13, 19, 25, 28, 29, 35, 37] which characterized compactness in fuzzy sets spaces equipped with different topologies.

Since Diamond and Kloeden [8] introduced $d_p$ metric which is a $L_p$-type metric, it has become one of the most often used convergence structure on fuzzy sets. Naturally, people have started to consider characterizations of compactness in fuzzy sets spaces with $d_p$ metric.

Diamond and Kloeden [8] gave compactness criteria for fuzzy number space $E^m$ with $d_p$ metric. Ma [25] modified the characterization given by Diamond and Kloeden. Convexity is a very useful property. Star-shapedness is a natural extension of convexity. Of cause, research on fuzzy counterparts of star-shaped sets has aroused the interest of people [5, 9]. Diamond [7] introduced the fuzzy star-shaped numbers as an extension of fuzzy numbers.

$S^m$ is used to denote the set of all fuzzy star-shaped numbers. Diamond [7] characterized the compact sets in $(S_0^m, d_p)$, where $S_0^m$ denotes the set of all the fuzzy star-shaped numbers with respect to the origin. $E^m$ and $S_0^m$ do not include each other. They both are subsets of $S^m$.

Wu and Zhao [35] pointed out that the compactness criteria for $(E^m, d_p)$ and $(S_0^m, d_p)$ in [7, 8] have the same type of error and that the modified compactness criteria in [25] still has fault. They [35] gave right characterizations of compactness in $(S_0^m, d_p)$ and $(E^m, d_p)$. Based on the results in [35], Zhao and Wu [37] further proposed a characterization of compactness in $(S^m, d_p)$. In these discussions, it is found that the concepts "$p$-mean equi-left-continuous" and "uniformly $p$-mean bounded" proposed by Diamond and Kloeden [8] and Ma [25], respectively, play an important role in establishing and illustrating characterizations of compactness in fuzzy sets spaces with $d_p$ metric.

Compare the characterizations in [35, 37] to Arzelà–Ascoli theorem, we find that the latter provides the compactness criteria by characterizing the totally bounded sets while the former does not seem to characterize the totally bounded sets. Since, in metric space, totally boundedness is a key feature of compactness, it is a natural and important problem to consider how to characterize totally bounded sets in fuzzy sets spaces with $d_p$ metric?

The existing three compactness criteria in [35, 37] are stated for three particular fuzzy sets spaces with $d_p$ metric. Fuzzy sets in these spaces have assumption of convexity or star-shapedness. It is worth noting that general fuzzy sets, which have no assumptions of convexity or star-shapedness, have attracted more and more attention in both theory and applications [2, 24, 36]. So this has caused a basic and important problem: how to characterize totally bounded sets, relatively compact set and compact sets in general fuzzy sets spaces endowed with $d_p$ metric?

There is another motivation to think about the problem above. Clearly, there exists other types of particular fuzzy sets. For example, Qiu et al.[27] introduced the set of all general fuzzy star-shaped numbers, which is denoted by $\widetilde{S}^m$. $S^m$ is a subset of $\widetilde{S}^m$ which in turn is a subset of $F_B(\mathbb{R}^m)$. If we characterize totally bounded sets, relatively compact sets and compact sets in general fuzzy sets spaces, then we can obtain parallel characterizations for particular fuzzy sets spaces immediately because the latter are subspaces of the former.

Analysis of Diamond [9] indicates that $(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$ and

$(\widetilde{S}^m, d_p)$ are not complete. Krätschmer [32] presented the completion of $(E^m, d_p)$ which is described by the support functions of fuzzy numbers. It is natural to consider a basic problem: what are the completions of all the rest spaces? Perhaps this question should be replaced by a more general question: how to generate the completions of fuzzy sets spaces with respect to $d_p$ metric? This problems are closely relevant to the problem of characterizing totally bounded sets and relatively compact sets in fuzzy sets spaces equipped with $d_p$ metric.

In this paper, we want to answer all questions above. These questions are closely relevant to each other. It can even be said that they are different aspects of a same problem. To put our discussion in a more general setting which does not have any assumptions of convexity or star-shapedness, we consider $F_B(\mathbb{R}^m)$ which is the set of all normal, upper semi-continuous, compact-support fuzzy sets on $\mathbb{R}^m$. Further we introduce the $L_p$-extension of a fuzzy sets space. The $L_p$-extensions of $(S_0^m, d_p)$, $(E^m, d_p)$, $(S^m, d_p)$, $(\widetilde{S}^m, d_p)$ and $(F_B(\mathbb{R}^m), d_p)$ are denoted by $(S_0^{m,p}, d_p)$, $(E^{m,p}, d_p)$, $(S^{m,p}, d_p)$, $(\widetilde{S}^{m,p}, d_p)$ and $(F_B(\mathbb{R}^m)^p, d_p)$, respectively. All the fuzzy sets spaces mentioned in this paper are subspaces of $(F_B(\mathbb{R}^m)^p, d_p)$. We give characterizations of totally bounded sets, relatively compact sets, and compact sets in $(F_B(\mathbb{R}^m), d_p)$ and $(F_B(\mathbb{R}^m)^p, d_p)$. Then it is proved that each $L_p$-extension mentioned in this paper is exactly the completion of its original space. Next, we show that the subspaces of $(F_B(\mathbb{R}^m), d_p)$ and $(F_B(\mathbb{R}^m)^p, d_p)$ have parallel characterizations of totally bounded sets, relatively compact sets, and compact sets to that of them. Finally, as applications of our results, we consider properties of $d_p$ metric and relook characterizations of compactness proposed in previous work.

The remainder part of this paper is organized as follows. Since $d_p$ metric is based on the well-known Hausdorff metric, Section 2 introduces and discusses some properties of Hausdorff metric. In Section 3, we recall and introduce some concepts and results of fuzzy sets related to our paper. Then, in Section 4, we present characterizations of relatively compact sets, totally bounded sets and compact sets in $(F_B(\mathbb{R}^m)^p, d_p)$. Section 5 shows that $F_B(\mathbb{R}^m)^p$ is in fact the completion of $F_B(\mathbb{R}^m)$ according to $d_p$ metric. Then we give characterizations of relatively compact sets, totally bounded sets and compact sets in $(F_B(\mathbb{R}^m), d_p)$. Based on the conclusions in Sections 4 and 5 and some discussions on convexity and star-shapedness of fuzzy sets, in Section 6, we show that $(S_0^{m,p}, d_p)$, $(E^{m,p}, d_p)$, $(S^{m,p}, d_p)$ and $(\widetilde{S}^{m,p}, d_p)$ are exactly the com-

pletions of $(S_0^m, d_p)$, $(E^m, d_p)$, $(S^m, d_p)$ and $(\widetilde{S}^m, d_p)$, respectively. We clarify relation among the ten fuzzy sets spaces discussed in this paper. As consequences of preceding results, it follows characterizations of totally bounded sets, relatively compact sets and compact sets in these spaces. Section 7 gives some applications of the results in our paper. At last, we draw conclusions in Section 8.

## 2. The Hausdorff metric

Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{Q}$ be the set of all rational numbers, $\mathbb{R}^m$ be $m$-dimensional Euclidean space, $K_C(\mathbb{R}^m)$ be the set of all the nonempty compact and convex sets in $\mathbb{R}^m$, $K(\mathbb{R}^m)$ be the set of all nonempty compact set in $\mathbb{R}^m$, and $C(\mathbb{R}^m)$ be the set of all nonempty closed set in $\mathbb{R}^m$. The well-known Hausdorff metric $H$ on $C(\mathbb{R}^m)$ is defined by:

$$H(U, V) = \max\{H^*(U, V),\ H^*(V, U)\}$$

for arbitrary $U, V \in C(\mathbb{R}^m)$, where

$$H^*(U, V) = \sup_{u \in U} d\,(u, V) = \sup_{u \in U} \inf_{v \in V} d\,(u, v).$$

**Proposition 2.1.** *[9] $(C(\mathbb{R}^m), H)$ is a complete metric space in which $K(\mathbb{R}^m)$ and $K_C(\mathbb{R}^m)$ are closed subsets. Hence, $K(\mathbb{R}^m)$ and $K_C(\mathbb{R}^m)$ are also complete metric spaces.*

**Proposition 2.2.** *[9, 28] A nonempty subset $U$ of $(K(\mathbb{R}^m), H)$ is compact if and only if it is closed and bounded in $(K(\mathbb{R}^m), H)$.*

**Proposition 2.3.** *[9] Let $\{u_n\} \subset K(\mathbb{R}^m)$ satisfy $u_1 \supseteq u_2 \supseteq \ldots \supseteq u_n \supseteq \ldots$. Then $u = \bigcap_{n=1}^{+\infty} u_n \in K(\mathbb{R}^m)$ and $H(u_n, u) \to 0$ as $n \to \infty$.*

*On the other hand, if $u_1 \subseteq u_2 \subseteq \ldots \subseteq u_n \subseteq \ldots$ and $u = \overline{\bigcup_{n=1}^{+\infty} u_n} \in K(\mathbb{R}^m)$, then $H(u_n, u) \to 0$ as $n \to \infty$.*

A set $K \in K(\mathbb{R}^m)$ is said to be star-shaped relative to a point $x \in K$ if for each $y \in K$, the line $\overline{xy}$ joining $x$ to $y$ is contained in $K$. The kernel $\ker K$ of $K$ is the set of all points $x \in K$ such that $\overline{xy} \subset K$ for each $y \in K$. The symbol $K_S(\mathbb{R}^m)$ is used to denote all the star-shaped sets in $\mathbb{R}^m$.

Obviously, $K_C(\mathbb{R}^m) \subsetneq K_S(\mathbb{R}^m)$. It can be checked that $\ker K \in K_C(\mathbb{R}^m)$ for all $K \in K_S(\mathbb{R}^m)$.

We say that a sequence of sets $\{C_n\}$ converges to $C$, in the sense of Kuratowski, if

$$C = \liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n,$$

where

$$\liminf_{n\to\infty} C_n = \{x \in X : \ x = \lim_{n\to\infty} x_n, x_n \in C_n\},$$

$$\limsup_{n\to\infty} C_n = \{x \in X : \ x = \lim_{j\to\infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} C_m}.$$

In this case, we'll write simply $C = \lim_{n\to\infty} C_n(K)$.

The following two known propositions discuss the relation of the convergence induced by Hausdorff metric and the convergence in the sense of Kuratowski. The readers can see [11] for details.

**Proposition 2.4.** *Suppose that $u$, $u_n$, $n = 1, 2, \ldots$, are nonempty compact sets in $\mathbb{R}^m$. Then $H(u_n, u) \to 0$ as $n \to \infty$ implies that $u = \lim_{n\to\infty} u_n(K)$.*

**Proposition 2.5.** *Suppose that $u$, $u_n$, $n = 1, 2, \ldots$, are nonempty compact sets in $\mathbb{R}^m$ and that $u_n$, $n = 1, 2, \ldots$, are connected sets. If $u = \lim_{n\to\infty} u_n(K)$, then $H(u_n, u) \to 0$ as $n \to \infty$.*

**Theorem 2.1.** *$K_S(\mathbb{R}^m)$ is a closed set in $(K(\mathbb{R}^m), H)$.*

**Proof.** Suppose that $\{u_n\} \subset K_S(\mathbb{R}^m)$, $u \in K(\mathbb{R}^m)$ and $H(u_n, u) \to 0$ as $n \to \infty$. In the following, we will prove that $u \in K_S(\mathbb{R}^m)$.

Choose $x_n \in \text{Ker } u_n$, $n = 1, 2, \ldots$, then there exists an $N$ such that $x_n \in U$ for all $n \geq N$, where $U := \{y : \ d(y, u) \leq 1\}$. Note that $U$ is a compact set, we know that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = x_0$. So $x_0 \in \limsup_{n\to\infty} u_n$, it then follows from Proposition 2.4 that $x_0 \in u$.

Now, we show that $u$ is star-shaped and $x_0 \in \ker u$. It suffices to show that

$$\lambda x_0 + (1 - \lambda)z \in u$$

for all $z \in u$ and $\lambda \in [0, 1]$. In fact, given $z \in u$, since $u = \liminf_{n\to\infty} u_n$, there is a sequence $\{z_n : z_n \in u_n\}$ such that $\lim_{n\to\infty} z_n = z$. Hence, for each $\lambda \in [0, 1]$,

$$\lambda x_0 + (1 - \lambda)z = \lim_{i\to\infty} \lambda x_{n_i} + (1 - \lambda)z_{n_i} \in \limsup_{n\to\infty} u_n,$$

and thus, by Proposition 2.4, $\lambda x_0 + (1 - \lambda)z \in u$. $\square$

**Corollary 2.1.** *Let $u$, $u_n$ be star-shaped sets, $n = 1, 2, \ldots$. If $H(u_n, u) \to 0$, then* $\limsup_{n\to\infty} \textit{ker}\, u_n \subset \textit{ker}\, u$.

**Proof.** From the proof of Theorem 2.1, we get the desired results. $\square$

**Remark 2.1.** We do not know whether Theorem 2.1 and Corollary 2.1 are known conclusions, so we give our proofs here.

## 3. The spaces of fuzzy sets

In this section, we recall and introduce various spaces of fuzzy sets including fuzzy numbers space, fuzzy star-shaped numbers space and general fuzzy star-shaped numbers space. Some basic properties of these spaces are discussed.

We use $F(\mathbb{R}^m)$ to represent all fuzzy subsets on $\mathbb{R}^m$, i.e. functions from $\mathbb{R}^m$ to $[0, 1]$. For details, we refer the readers to references [9, 34]. $\mathbf{2}^{\mathbb{R}^m} := \{S : S \subseteq \mathbb{R}^m\}$ can be embedded in $F(\mathbb{R}^m)$, as any $S \subset \mathbb{R}^m$ can be seen as its characteristic function, i.e. the fuzzy set

$$\widehat{S}(x) = \begin{cases} 1, x \in S, \\ 0, x \notin S. \end{cases}$$

For $u \in F(\mathbb{R}^m)$, let $[u]_\alpha$ denote the $\alpha$-cut of $u$, i.e.

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R}^m : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \operatorname{supp} u = \overline{\{x \in \mathbb{R}^m : u(x) > 0\}}, & \alpha = 0. \end{cases}$$

For $u \in F(\mathbb{R}^m)$, we suppose that
(i) $u$ is normal: there exists at least one $x_0 \in \mathbb{R}^m$ with $u(x_0) = 1$;
(ii) $u$ is upper semi-continuous;
(iii-1) $u$ is fuzzy convex: $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in \mathbb{R}^m$ and $\lambda \in [0, 1]$;
(iii-$2^0$) $u$ is fuzzy star-shaped with respect to the origin, i.e., $u(\lambda y) \geq u(y)$ for all $y \in \mathbb{R}^m$ and $\lambda \in [0, 1]$.
(iii-2) $u$ is fuzzy star-shaped, i.e., there exists $x \in \mathbb{R}^m$ such that $u$ is fuzzy star-shaped with respect to $x$, namely, $u(\lambda y + (1 - \lambda)x) \geq u(y)$ for all $y \in \mathbb{R}^m$ and $\lambda \in [0, 1]$;
(iii-3) Given $\alpha \in (0, 1]$, then there exists $x_\alpha \in [u]_\alpha$ such that $\overline{x_\alpha\, y} \in [u]_\alpha$ for all $y \in [u]_\alpha$;

(iv-1) $[u]_0$ is a bounded set in $\mathbb{R}^m$;
(iv-2) $\left(\int_0^1 H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty$, where $p \geq 1$ and 0 denotes the origin of $\mathbb{R}^m$;
(iv-3) $[u]_\alpha$ is a bounded set in $\mathbb{R}^m$ when $\alpha > 0$.

- If $u$ satisfies (i), (ii), (iii-1) and (iv-1), then $u$ is a fuzzy number. The set of all fuzzy numbers is denoted by $E^m$.

- If $u$ satisfies (i), (ii), (iii-$2^0$) and (iv-1), then we say $u$ is a fuzzy star-shaped number with respect to the origin. The set of all fuzzy star-shaped numbers with respect to the origin is denoted by $S_0^m$.

- If $u$ satisfies (i), (ii), (iii-2) and (iv-1), then we say $u$ is a fuzzy star-shaped number. The set of all fuzzy star-shaped numbers is denoted by $S^m$.

- If $u$ satisfies (i), (ii), (iii-3) and (iv-1), then we say $u$ is a general fuzzy star-shaped number. The set of all general fuzzy star-shaped numbers is denoted by $\widetilde{S}^m$.

The definitions of fuzzy star-shaped numbers and general fuzzy star-shaped numbers were given by Diamond [9] and Qiu et al. [27], respectively. $\mathbb{R}^m$ can be embedded in $E^m$, as any $r \in \mathbb{R}^m$ can be viewed as the fuzzy number

$$\widehat{r}(x) = \begin{cases} 1, x = r, \\ 0, x \neq r. \end{cases}$$

We can see that $E^m \nsubseteq S_0^m$ and $S_0^m \nsubseteq E^m$. If $u \in S^m$, then $\bigcap_{\alpha\in(0,1]} \ker [u]_\alpha \neq \emptyset$, however this inequality may not hold when $u \in \widetilde{S}^m$. Clearly $S_0^m = \{u \in S^m : \ 0 \in \bigcap_{\alpha\in(0,1]} \ker [u]_\alpha\}$. So $E^m, S_0^m \subsetneqq S^m \subsetneqq \widetilde{S}^m$.

In order to illustrate and prove the conclusions in this paper, we need to introduce $L_p$–type noncompact fuzzy sets.

Let $u \in F(\mathbb{R}^m)$.

- If $u$ satisfies (i), (ii), (iii-1) and (iv-2), then we say $u$ is a $L_p$-type noncompact fuzzy number. The collection of all such fuzzy sets is denoted by $E^{m,p}$.

- If $u$ satisfies (i), (ii), (iii-$2^0$) and (iv-2), then we say $u$ is a $L_p$-type noncompact fuzzy star-shaped number with respect to the origin. The collection of all such fuzzy sets is denoted by $S_0^{m,p}$.

- If $u$ satisfies (i), (ii), (iii-2) and (iv-2), then we say $u$ is a $L_p$-type noncompact fuzzy star-shaped number. The collection of all such fuzzy sets is denoted by $S^{m,p}$.

- If $u$ satisfies (i), (ii), (iii-3) and (iv-2), then we say $u$ is a $L_p$-type noncompact general fuzzy star-shaped number. The collection of all such fuzzy sets is denoted by $\widetilde{S}^{m,p}$.

It's easy to check that $E^m \subsetneqq E^{m,p}$, $S_0^m \subsetneqq S_0^{m,p}$, $S^m \subsetneqq S^{m,p}$ and $\widetilde{S}^m \subsetneqq \widetilde{S}^{m,p}$.

We can see a kind of $L_p$–type noncompact fuzzy sets is obtained by using weaker assumption (iv-2) to replace stronger assumption (iv-1) on the corresponding kind of compact fuzzy sets. So the latter is a subset of the former. We call this process *$L_p$-extension*. A kind of $L_p$–type noncompact fuzzy sets is called *$L_p$-extension* of the corresponding kind of compact fuzzy sets.

Above eight kinds of fuzzy sets have assumptions of convexity or star-shapedness. In recent years, people pay more and more attention to general fuzzy sets from points of view of theoretical research and real-world applications. For example, in the study of fuzzy differential equations [2, 24], researchers consider fuzzy sets with no assumptions of convexity or star-shapedness. For this reason, we wish discussions in this paper can be put in a setting of general fuzzy sets which have no assumption of convexity or star-shapedness. So we introduce the following kinds of general fuzzy sets.

Suppose $u \in F(\mathbb{R}^m)$.

- If $u$ satisfies (i), (ii) and (iv-1), then $u$ is a normal upper semi-continuous compact-support fuzzy set on $\mathbb{R}^m$. The collection of all such fuzzy sets is denoted by $F_B(\mathbb{R}^m)$.

- If $u$ satisfies (i), (ii) and (iv-2), then $u$ is a normal upper semi-continuous $L_p$-type noncompact-support fuzzy set on $\mathbb{R}^m$. The collection of all such fuzzy sets is denoted by $F_B(\mathbb{R}^m)^p$.

- If $u$ satisfies (i), (ii) and (iv-3), then $u$ is a normal upper semi-continuous noncompact-support fuzzy set on $\mathbb{R}^m$. The collection of all such fuzzy sets is denoted by $F_{GB}(\mathbb{R}^m)$.

Clearly, $F_B(\mathbb{R}^m)^p$ is the $L_p$-extension of $F_B(\mathbb{R}^m)$. We can check that

$$E^m, S_0^m \subsetneqq S^m \subsetneqq \widetilde{S}^m \subsetneqq F_B(\mathbb{R}^m),$$
$$E^{m,p}, S_0^{m,p} \subsetneqq S^{m,p} \subsetneqq \widetilde{S}^{m,p} \subsetneqq F_B(\mathbb{R}^m)^p,$$

$$F_B(\mathbb{R}^m) \subsetneq F_B(\mathbb{R}^m)^p \subsetneq F_{GB}(\mathbb{R}^m).$$

Diamond and Kloeden [9] introduced the $d_p$ distance $(1 \leq p < \infty)$ on $S^m$ which is defined by

$$d_p(u,v) = \left( \int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p} \tag{1}$$

for all $u, v \in S^m$. Note that $u \in F(\mathbb{R}^m)$ satisfies assumption (ii) is equivalent to $[u]_\alpha \in C(\mathbb{R}^m)$ for all $\alpha \in (0,1]$. So $d_p$ distance $(1 \leq p < \infty)$ can be defined on $F_{GB}(\mathbb{R}^m)$. But $d_p$ distance is not a metric on $F_{GB}(\mathbb{R}^m)$ because $d_p(u,v)$ may equal $+\infty$ for some $u, v \in F_{GB}(\mathbb{R}^m)$. It's easy to check that $d_p$ distance, $p \geq 1$, is a metric on $F_B(\mathbb{R}^m)^p$. All the fuzzy sets spaces mentioned in this paper are subspaces of $(F_B(\mathbb{R}^m)^p, d_p)$.

A kind of $L_p$–type noncompact fuzzy sets space endowed with $d_p$ metric is called $L_p$-*extension* of the corresponding compact fuzzy sets space with $d_p$ metric. So fuzzy sets spaces $(E^{m,p}, d_p)$, $(S_0^{m,p}, d_p)$, $(S^{m,p}, d_p)$, $(\widetilde{S}^{m,p}, d_p)$ and $(F_B(\mathbb{R}^m)^p, d_p)$ are the $L_p$-extensions of fuzzy sets spaces $(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$, $(\widetilde{S}^m, d_p)$ and $(F_B(\mathbb{R}^m), d_p)$, respectively.

Diamond and Kloeden [9] pointed out that $(E^m, d_p)$ is not a complete space. Their analysis also indicates that the four spaces $(S_0^m, d_p)$, $(S^m, d_p)$, $(\widetilde{S}^m, d_p)$ and $(F_B(\mathbb{R}^m), d_p)$ are not complete. Krätschmer [32] has given the completion of $(E^m, d_p)$ which is described by using support functions of fuzzy numbers.

In the sequel of this paper, we show that the completion of every incomplete fuzzy sets space mentioned in this paper is exactly its $L_p$-extension , i.e., their completions can be obtained by means of $L_p$-extension.

**Remark 3.1.** It can be checked that for each $u \in F_B(\mathbb{R}^m)^p$, $[u]_\alpha$ is a compact set when $\alpha \in (0,1]$, and $[u]_0$, the 0-cut, is the only possible unbounded cut-set. So we know that $[u]_\alpha \in K_C(\mathbb{R}^m)$ for all $u \in E^{m,p}$ and $\alpha \in (0,1]$, and that $[u]_\alpha \in K_S(\mathbb{R}^m)$ for all $u \in \widetilde{S}^{m,p}$ and $\alpha \in (0,1]$.

Denote ker $u := \bigcap_{\alpha \in (0,1]} \ker\, [u]_\alpha$ for $u \in \widetilde{S}^{m,p}$ (also see [9, 27]). It is easy to check that, given $u \in \widetilde{S}^{m,p}$, then $u \in S^{m,p}$ if and only if ker $u \neq \emptyset$.

The following representation theorem is used widely in the theory of fuzzy numbers.

**Proposition 3.1.** *[26] Given $u \in E^m$, then*
*(i)* $[u]_\lambda \in K_C(\mathbb{R}^m)$ *for all* $\lambda \in [0,1]$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iii)$ *then there exists a unique* $u \in E^m$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

Similarly, we can obtain representation theorems for $S_0^m$, $S^m$, $\widetilde{S}^m$, $F_B(\mathbb{R}^m)$, $E^{m,p}$, $S_0^{m,p}$, $S^{m,p}$, $\widetilde{S}^{m,p}$ and $F_B(\mathbb{R}^m)^p$ which are listed below and will be used in the sequel.

**Theorem 3.1.** *Given* $u \in S_0^m$*, then*
*(i)* $[u]_\lambda \in K_S(\mathbb{R}^m)$ *for all* $\lambda \in [0,1]$*, and* $0 \in \bigcap_{\alpha\in(0,1]} \mathit{ker}\, [u]_\alpha$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iii)$*, then there exists a unique* $u \in S_0^m$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.2.** *Given* $u \in S^m$*, then*
*(i)* $[u]_\lambda \in K_S(\mathbb{R}^m)$ *for all* $\lambda \in [0,1]$*, and* $\bigcap_{\alpha\in(0,1]} \mathit{ker}\, [u]_\alpha \neq \emptyset$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iii)$*, then there exists a unique* $u \in S^m$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.3.** *Given* $u \in \widetilde{S}^m$*, then*
*(i)* $[u]_\lambda \in K_S(\mathbb{R}^m)$ *for all* $\lambda \in [0,1]$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iii)$*, then there exists a unique* $u \in \widetilde{S}^m$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.4.** *Given* $u \in F_B(\mathbb{R}^m)$*, then*
*(i)* $[u]_\lambda \in K(\mathbb{R}^m)$ *for all* $\lambda \in [0,1]$*;*

*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iii)$*, then there exists a unique* $u \in F_B(\mathbb{R}^m)$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.5.** *Given* $u \in E^{m,p}$*, then*
*(i)* $[u]_\lambda \in K_C(\mathbb{R}^m)$ *for all* $\lambda \in (0,1]$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$*;*
*(iv)* $\left(\int_0^1 H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iv)$*, then there exists a unique* $u \in E^{m,p}$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.6.** *Given* $u \in S_0^{m,p}$*, then*
*(i)* $[u]_\lambda \in K_S(\mathbb{R}^m)$ *for all* $\lambda \in (0,1]$ *and* $0 \in \bigcap_{\alpha\in(0,1]} ker\,[u]_\alpha$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$*;*
*(iv)* $\left(\int_0^1 H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iv)$*, then there exists a unique* $u \in S_0^{m,p}$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.7.** *Given* $u \in S^{m,p}$*, then*
*(i)* $[u]_\lambda \in K_S(\mathbb{R}^m)$ *for all* $\lambda \in (0,1]$ *and* $\bigcap_{\alpha\in(0,1]} ker\,[u]_\alpha \neq \emptyset$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$*;*
*(iv)* $\left(\int_0^1 H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iv)$*, then there exists a unique* $u \in S^{m,p}$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.8.** *Given* $u \in \widetilde{S}^{m,p}$*, then*
*(i)* $[u]_\lambda \in K_S(\mathbb{R}^m)$ *for all* $\lambda \in (0,1]$*;*
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$*;*

*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$;

*(iv)* $\left(\int_0^1 H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iv)$, *then there exists a unique* $u \in \widetilde{S}^{m,p}$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

**Theorem 3.9.** *Given* $u \in F_B(\mathbb{R}^m)^p$, *then*
*(i)* $[u]_\lambda \in K(\mathbb{R}^m)$ *for all* $\lambda \in (0,1]$;
*(ii)* $[u]_\lambda = \bigcap_{\gamma<\lambda}[u]_\gamma$ *for all* $\lambda \in (0,1]$;
*(iii)* $[u]_0 = \overline{\bigcup_{\gamma>0}[u]_\gamma}$;
*(iv)* $\left(\int_0^1 H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty$.

*Moreover, if the family of sets* $\{v_\alpha : \alpha \in [0,1]\}$ *satisfy conditions* $(i)$ *through* $(iv)$, *then there exists a unique* $u \in F_B(\mathbb{R}^m)^p$ *such that* $[u]_\lambda = v_\lambda$ *for each* $\lambda \in [0,1]$.

## 4. Characterizations of relatively compact sets, totally bounded sets and compact sets in $(F_B(\mathbb{R}^m)^p, d_p)$

In this section, we present a characterization of relatively compact sets in fuzzy sets space $(F_B(\mathbb{R}^m)^p, d_p)$. Based on this, we then give characterizations of totally bounded sets and compact sets in $(F_B(\mathbb{R}^m)^p, d_p)$.

The topic of characterizations of compactness of fuzzy sets spaces with $d_p$ metric has been studied for many years. The following two concepts are important to establish and illustrate characterizations of compactness, which are introduced by Diamond and Kloeden [8] and Ma [25].

**Definition 4.1.** *[8] Let* $u \in F_B(\mathbb{R}^m)^p$. *If for given* $\varepsilon > 0$, *there is a* $\delta(u,\varepsilon) > 0$ *such that for all* $0 \le h < \delta$

$$\left(\int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha\right)^{1/p} < \varepsilon,$$

*where* $1 \le p < +\infty$, *then we say* $u$ *is p-mean left-continuous.*

*Suppose that* $U$ *is a nonempty set in* $F_B(\mathbb{R}^m)^p$. *If the above inequality holds uniformly for all* $u \in U$, *then we say* $U$ *is p-mean equi-left-continuous.*

**Definition 4.2.** *[25] A set* $U \subset F_B(\mathbb{R}^m)^p$ *is said to be uniformly p-mean bounded if there is a constant* $M > 0$ *such that* $d_p(u, \widehat{0}) \le M$ *for all* $u \in U$.

It is easy to check that $U$ is uniformly $p$-mean bounded is equivalent to $U$ is a bounded set in $(F_B(\mathbb{R}^m)^p, d_p)$.

Diamond [7, 8] characterized the compact sets in $(E^m, d_p)$ and $(S_0^m, d_p)$.

**Proposition 4.1.** *[8] A closed set $U$ of $(E^m, d_p)$ is compact if and only if:*
*(i) $\{[u]_0 : u \in U\}$ is bounded in $(K(\mathbb{R}^m), H)$;*
*(ii) $U$ is p-mean equi-left-continuous.*

**Proposition 4.2.** *[7] A closed set $U$ of $(S_0^m, d_p)$ is compact if and only if:*
*(i) $\{[u]_0 : u \in U\}$ is bounded in $(K(\mathbb{R}^m), H)$;*
*(ii) $U$ is p-mean equi-left-continuous.*

Ma [25] found that there exists an error in Proposition 4.1 and modified it to following Proposition 4.3. For the convenience of writing, let $u \in F(\mathbb{R}^m)$, Ma used the symbol $u^{(\alpha)}$ to denote the fuzzy set

$$u^{(\alpha)}(x) := \begin{cases} u(x), & \text{if } u(x) \geq \alpha, \\ 0, & \text{if } u(x) < \alpha. \end{cases}$$

**Proposition 4.3.** *[25] A closed set $U$ of $(E^m, d_p)$ is compact if and only if:*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous;*
*(iii) For $\{u_k\} \subset U$, if $\{u_k^{(h)} : k = 1, 2, \ldots\}$ converges to $u(h) \in E^m$ in $d_p$ metric for any $h > 0$, then there exists a $u_0 \in E^m$ such that $u_0^{(h)} = u(h)$.*

Wu and Zhao [35] pointed out there still exists fault in compactness characterization given by Ma [25]. They showed that Propositions 4.1, 4.2 and 4.3 are all wrong and then gave right forms of compactness criteria for $(E^m, d_p)$ and $(S_0^m, d_p)$.

**Proposition 4.4.** *[35] A closed set $U$ of $(E^m, d_p)$ is compact if and only if:*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous;*
*(iii) Let $r_i$ be a decreasing sequence in $(0, 1]$ converging to zero. For $\{u_k\} \subset U$, if $\{u_k^{(r_i)} : k = 1, 2, \ldots\}$ converges to $u(r_i) \in E^m$ in $d_p$ metric, then there exists a $u_0 \in E^m$ such that $[u_0^{(r_i)}]_\alpha = [u(r_i)]_\alpha$ when $r_i < \alpha \leq 1$.*

**Proposition 4.5.** *[35] A closed set $U$ of $(S_0^m, d_p)$ is compact if and only if:*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous;*

*(iii) Let $r_i$ be a decreasing sequence in $(0,1]$ converging to zero. For $\{u_k\} \subset U$, if $\{u_k^{(r_i)} : k = 1, 2, \ldots\}$ converges to $u(r_i) \in S_0^m$ in $d_p$ metric, then there exists a $u_0 \in S_0^m$ such that $[u_0^{(r_i)}]_\alpha = [u(r_i)]_\alpha$ when $r_i < \alpha \leq 1$.*

Based on Proposition 4.5, Zhao and Wu [37] further presented compactness criteria of $(S^m, d_p)$.

**Proposition 4.6.** *[37] A closed set $U$ of $(S^m, d_p)$ is compact if and only if:*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous;*
*(iii) Let $r_i$ be a decreasing sequence in $(0,1]$ converging to zero. For $\{u_k\} \subset U$, if $\{u_k^{(r_i)} : k = 1, 2, \ldots\}$ converges to $u(r_i) \in S^m$ in $d_p$ metric, then there exists a $u_0 \in S^m$ such that $[u_0^{(r_i)}]_\alpha = [u(r_i)]_\alpha$ when $r_i < \alpha \leq 1$.*

Compare Propositions 4.4, 4.5, 4.6 with Arzelà–Ascoli theorem, we notice that the latter provides the compactness criteria by characterizing the totally bounded set while the former does not seem to do so. Since totally boundedness is a key feature of compactness, it is a basic and important problem to consider characterizations of totally bounded sets in fuzzy sets spaces. In order to obtain characterization of totally bounded sets in $(F_B(\mathbb{R}^m)^p, d_p)$, we first give a characterization of relatively compact sets in $(F_B(\mathbb{R}^m)^p, d_p)$.

Some fundamental conclusions and concepts in classic analysis and topology are listed below, which are useful in this paper. The readers can see [3] for details.

- **Lebesgue's Dominated Convergence Theorem**. Let $\{f_n\}$ be a sequence of integrable functions that converges almost everywhere to a function $f$, and suppose that $\{f_n\}$ is dominated by an integrable function $g$. Then $f$ is integrable, and $\int f = \lim_{n\to\infty} \int f_n$.

- **Fatou's Lemma**. Let $\{f_n\}$ be a sequence of nonnegative integrable functions that converges almost everywhere to a function $f$, and if the sequence $\{\int f_n\}$ is bounded above, then $f$ is integrable and $\int f \leq \liminf \int f_n$

- **Absolute continuity of Lebesgue integral**. Suppose that $f$ is Lebesgue integrable on $E$, then for arbitrary $\varepsilon > 0$, there is a $\delta > 0$ such that $\int_A f\, dx < \varepsilon$ whenever $A \subseteq E$ and $m(A) < \delta$.

- **Minkowski's inequality**. Let $p \geq 1$, and let $f, g$ be measurable functions on $\mathbb{R}$ such that $|f|^p$ and $|g|^p$ are integrable. Then $|f+g|^p$ is integrable, and Minkowski's inequality

$$\left(\int |f+g|^p\right)^{1/p} \leq \left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}$$

holds.

- A relatively compact subset $Y$ of a topological space X is a subset whose closure is compact. In the case of a metric topology, the criterion for relative compactness becomes that any sequence in $Y$ has a subsequence convergent in $X$.

- Let $(X, d)$ be a metric space. A set $U$ in $X$ is totally bounded if and only if for each $\varepsilon > 0$, it contains a finite $\varepsilon$-approximation, where an $\varepsilon$-approximation to $U$ is a subset $S$ of $U$ such that $\rho(x, S) < \varepsilon$ for each $x \in U$.

- Let $(X, d)$ be a metric space. Then a set $U$ in $X$ is relatively compact implies that it is totally bounded. For subsets of a complete metric space these two meanings coincide. Thus $(X, d)$ is a compact space iff $X$ is totally bounded and complete.

We start with some lemmas which discuss some properties of bounded subsets and elements of $(F_B(\mathbb{R}^m)^p, d_p)$.

**Lemma 4.1.** *Suppose that $U$ is a bounded set in $(F_B(\mathbb{R}^m)^p, d_p)$, then $\{[u]_\alpha : u \in U\}$ is a bounded set in $(K(\mathbb{R}^m), H)$ for each $\alpha > 0$.*

**Proof.** If there exists an $\alpha_0 > 0$ such that $\{[u]_{\alpha_0} : u \in U\}$ is not a bounded set in $(K(\mathbb{R}^m), H)$. Then there is a $u \in U$ such that $[u]_{\alpha_0} \notin K(\mathbb{R}^m)$ or $\{[u]_{\alpha_0} : u \in U\}$ is a unbounded set in $K(\mathbb{R}^m)$. For both cases, there exist $u_n \in U$ such that $H([u_n]_{\alpha_0}, \{0\}) > n \cdot (\frac{1}{\alpha_0})^{1/p}$ when $n = 1, 2, \ldots$, and hence $\left(\int_0^1 H([u_n]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} > n$, which contradicts the boundness of $U$. □

**Lemma 4.2.** *Suppose that $u \in F_B(\mathbb{R}^m)^p$ and $\alpha \in (0, 1]$, then $H([u]_\alpha, [u]_\beta) \to 0$ as $\beta \to \alpha-$.*

**Proof.** The desired result follows immediately from Theorem 3.9 and Proposition 2.3. □

**Lemma 4.3.** *If $u \in F_B(\mathbb{R}^m)^p$, then $u$ is $p-$mean left-continuous.*

**Proof.** Given $\varepsilon > 0$. Note that $u \in F_B(\mathbb{R}^m)^p$, from the absolute continuity of Lebesgue integral, we know there exists an $h_1 > 0$ such that

$$\left( \int_0^{h_1} H([u]_\alpha, \{0\})^p \, d\alpha \right)^{1/p} \leq \varepsilon/3. \tag{2}$$

Note that $[u]_{\frac{h_1}{2}} \in K(\mathbb{R}^m)$, then there exists an $M > 0$ such that $H([u]_{\frac{h_1}{2}}, \{0\}) < M$. This yields that

$$H([u]_\alpha, [u]_{\alpha-h}) \leq H([u]_\alpha, \{0\}) + H([u]_{\alpha-h}, \{0\}) \leq 2M \tag{3}$$

for all $\alpha \geq h_1$ and $h \leq h_1/2$.

By Lemma 4.2,we know that $H([u]_\alpha, [u]_{\alpha-h}) \to 0$ when $h \to 0+$, it then follows from the Lebesgue's dominated convergence theorem and (3) that

$$\left( \int_{h_1}^{1} H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \to 0$$

when $h \to 0+$. Thus there exists an $h_2 > 0$ such that

$$\left( \int_{h_1}^{1} H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} < \varepsilon/3 \tag{4}$$

for all $0 \leq h \leq \min\{h_1/2, h_2\}$.

Now combined (2) and (4), we know that, for all $h \leq h_3 = \min\{h_1/2, h_2\}$,

$$\begin{aligned}
&\left( \int_{h}^{1} H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \\
&\leq \left( \int_{h}^{h_1} H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} + \left( \int_{h_1}^{1} H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \\
&\leq \left( \int_{h}^{h_1} H([u]_\alpha, \{0\})^p \, d\alpha \right)^{1/p} + \left( \int_{h}^{h_1} H([u]_{\alpha-h}, \{0\})^p \, d\alpha \right)^{1/p} + \varepsilon/3 \\
&\leq \left( \int_{0}^{h_1} H([u]_\alpha, \{0\})^p \, d\alpha \right)^{1/p} + \left( \int_{0}^{h_1} H([u]_\alpha, \{0\})^p \, d\alpha \right)^{1/p} + \varepsilon/3 \\
&\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned} \tag{5}$$

From the arbitrariness of $\varepsilon$, we know that $u$ is $p-$mean left-continuous. $\square$

Now, we arrive at the main results of this section. The following theorem gives a characterization of relatively compactness in $(F_B(\mathbb{R}^m)^p, d_p)$.

**Theorem 4.1.** *$U$ is a relatively compact set in $(F_B(\mathbb{R}^m)^p, d_p)$ if and only if*
*(i) $U$ is uniformly $p$-mean bounded;*
*(ii) $U$ is $p$-mean equi-left-continuous.*

**Proof.** ***Necessity.*** If $U$ is a relatively compact set in $(F_B(\mathbb{R}^m)^p, d_p)$. Since $(F_B(\mathbb{R}^m)^p, d_p)$ is a metric space, it follows immediately that $U$ is a bounded set in $(F_B(\mathbb{R}^m)^p, d_p)$, i.e. $U$ is uniformly $p$-mean bounded.

Now we prove that $U$ is $p-$mean equi-left-continuous. Given $\varepsilon > 0$. Since $U$ is a relatively compact set, there exists an $\varepsilon/3$-net $\{u_1, u_2, \dots u_n\}$ of $U$. From Lemma 4.3, we know that $\{u_k : k = 1, 2, \dots n\}$ is $p$-mean equi-left-continuous. Hence there exists $\delta > 0$ such that

$$\left( \int_h^1 H([u_k]_\alpha, [u_k]_{\alpha-h})^p \, d\alpha \right)^{1/p} \leq \varepsilon/3. \tag{6}$$

for all $h \in [0, \delta)$ and $k = 1, 2, \dots, n$.

Given $u \in U$, there is an $u_k$ such that $d_p(u, u_k) \leq \varepsilon/3$, and thus, by (6), we know that for all $h \in [0, \delta)$,

$$\begin{aligned}
&\left( \int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \\
&\leq \left( \int_h^1 H([u]_\alpha, [u_k]_\alpha)^p \, d\alpha \right)^{1/p} + \left( \int_h^1 H([u_k]_\alpha, [u_k]_{\alpha-h})^p \, d\alpha \right)^{1/p} + \left( \int_h^1 H([u_k]_{\alpha-h}, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \\
&\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned} \tag{7}$$

From the arbitrariness of $\varepsilon$ and $u \in U$, we obtain that $U$ is $p$-mean equi-left-continuous.

***Sufficiency.*** If $U$ satisfies (i) and (ii). To show $U$ is a relatively compact set, it suffices to find a convergent subsequence of an arbitrarily given sequence in $U$.

Let $\{u_n\}$ be a sequence in $U$. To find a subsequence $\{v_n\}$ of $\{u_n\}$ which converges to $v \in F_B(\mathbb{R}^m)^p$ according to $d_p$ metric, we split the proof into three steps.

**Step 1.** Find a subsequence $\{v_n\}$ of $\{u_n\}$ and $v \in F_{GB}(\mathbb{R}^m)$ such that

$$H([v_n]_\alpha, [v]_\alpha) \stackrel{\text{a.e.}}{\to} 0 \ ([0,1]). \tag{8}$$

Since $U$ is uniformly $p-$mean bounded, by Lemma 4.1, $\{[u]_\alpha : u \in U\}$ is a bounded set in $K(\mathbb{R}^m)$ for each $\alpha \in (0,1]$. Thus, by Proposition 2.2, for each $\alpha > 0$, $\{[u]_\alpha : u \in U\}$ is a relatively compact set in $(K(\mathbb{R}^m), H)$.

Arrange all rational numbers in $(0,1]$ into a sequence $q_1, q_2, \ldots, q_n, \ldots$. Then $\{u_n\}$ has a subsequence $\{u_n^{(1)}\}$ such that $\{[u_n^{(1)}]_{q_1}\}$ converges to $u_{q_1} \in K(\mathbb{R}^m)$, i.e. $H([u_n^{(1)}]_{q_1}, u_{q_1}) \to 0$. If $\{u_n^{(1)}\}, \ldots, \{u_n^{(k)}\}$ have been chosen, we can choose a subsequence $\{u_n^{(k+1)}\}$ of $\{u_n^{(k)}\}$ such that $\{[u_n^{(k+1)}]_{q_{k+1}}\}$ converges to $u_{q_{k+1}} \in K(\mathbb{R}^m)$. Thus we obtain nonempty compact sets $u_{q_k}, k = 1, 2, \ldots$. with $u_{q_m} \subseteq u_{q_l}$ whenever $q_m > q_l$.

Put $v_n = \{u_n^{(n)}\}$ for $n = 1, 2, \ldots$. Then $\{v_n\}$ is a subsequence of $\{u_n\}$ and

$$H([v_n]_{q_k}, u_{q_k}) \to 0 \quad \text{as} \quad n \to \infty \tag{9}$$

for $k = 1, 2, \ldots$. Define $\{v_\alpha : \alpha \in [0,1]\}$ as follows:

$$v_\alpha = \begin{cases} \bigcap_{q_k < \alpha} u_{q_k}, & \alpha \in (0,1]; \\ \overline{\bigcup_{\alpha \in (0,1]} v_\alpha}, & \alpha = 0. \end{cases}$$

Then $v_\alpha$, $\alpha \in [0,1]$, have the following properties:

(i) $v_\lambda \in K(\mathbb{R}^m)$ for all $\lambda \in (0,1]$;

(ii) $v_\lambda = \bigcap_{\gamma < \lambda} v_\gamma$ for all $\lambda \in (0,1]$;

(iii) $v_0 = \overline{\bigcup_{\gamma > 0} v_\gamma}$.

In fact, by Proposition 2.3, we obtain that $v_\alpha \in K(\mathbb{R}^m)$ for all $\alpha \in (0,1]$. Thus property (i) is proved. Properties (ii) and (iii) follow immediately from the definition of $v_\alpha$.

Define a function $v : \mathbb{R}^m \to [0,1]$ by

$$v(x) = \begin{cases} \bigvee_{x \in v_\lambda} \lambda, & x \in \bigcup_{\lambda > 0} v_\lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Then $v$ is a fuzzy set on $\mathbb{R}^m$. From properties (i), (ii) and (iii) of $v_\alpha$, we know that

$$[v]_\alpha = v_\alpha.$$

So $v \in F_{GB}(\mathbb{R}^m)$. Clearly if the following statements (I) and (II) are true, then we obtain (8), i.e. $H([v_n]_\alpha, [v]_\alpha) \xrightarrow{\text{a.e.}} 0$ $([0,1])$.

(I) $P(v)$ is at most countable, where $P(v) = \{\alpha \in (0,1) : \overline{\{v > \alpha\}} \subsetneq [v]_\alpha\}$, where $\overline{\{v > \alpha\}} := \overline{\bigcup_{\beta>\alpha}[v]_\beta}$.

(II) If $\alpha \in (0,1)\backslash P(v)$, then

$$H([v_n]_\alpha, [v]_\alpha) \to 0 \text{ as } n \to \infty. \tag{10}$$

Firstly, we show assertion (I). Let $D(v) := \{\alpha \in (0,1) : [v]_\alpha \nsubseteq \overline{\{v > \alpha\}}\ \}$. Notice that $P(v) \subseteq D(v)$ (In fact, it can be checked that $P(v) = D(v)$). By the conclusion in Appendix of [21], $D(v)$ is at most countable. So $P(v)$ is at most countable.

Secondly, we show assertion (II). Suppose that $\alpha \in (0,1)\backslash P(v)$, then from Proposition 2.3, $H([v]_\beta, [v]_\alpha) \to 0$ as $\beta \to \alpha$. Thus, given $\varepsilon > 0$, we can find a $\delta > 0$ such that $H(u_q, v_\alpha) < \varepsilon$ for all $q \in \mathbb{Q}$ with $|q - \alpha| < \delta$. So

$$H^*([v_n]_\alpha, v_\alpha) \le H^*([v_n]_{q_1}, v_\alpha) \le H^*([v_n]_{q_1}, u_{q_1}) + \varepsilon$$

for $q_1 \in \mathbb{Q} \cap (\alpha - \delta, \alpha)$. Hence, by (9) and the arbitrariness of $\varepsilon$, we obtain

$$H^*([v_n]_\alpha, v_\alpha) \to 0 \ (n \to \infty). \tag{11}$$

On the other hand,

$$H^*(v_\alpha, [v_n]_\alpha) \le H^*(v_\alpha, [v_n]_{q_2}) \le H^*(u_{q_2}, [v_n]_{q_2}) + \varepsilon$$

for $q_2 \in \mathbb{Q} \cap (\alpha, \alpha + \delta)$. Hence, by (9) and the arbitrariness of $\varepsilon$, we obtain

$$H^*(v_\alpha, [v_n]_\alpha) \to 0 \ (n \to \infty). \tag{12}$$

Combined with (11) and (12), we thus obtain (10).

**Step 2.** Prove that

$$\left(\int_0^1 H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha\right)^{1/p} \to 0. \tag{13}$$

Given $\varepsilon > 0$. It can be deduced that, for all $h < 1/2$,

$$\begin{aligned}
&\left(\int_0^h H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha\right)^{1/p} \\
\le &\left(\int_0^h H([v_n]_\alpha, [v_n]_{\alpha+h})^p \, d\alpha\right)^{1/p} + \left(\int_0^h H([v_n]_{\alpha+h}, [v]_{\alpha+h})^p \, d\alpha\right)^{1/p} + \left(\int_0^h H([v]_{\alpha+h}, [v]_\alpha)^p \, d\alpha\right)^{1/p}
\end{aligned}$$

$$= \left( \int_h^{2h} H([v_n]_{\beta-h}, [v_n]_\beta)^p \, d\beta \right)^{1/p} + \left( \int_h^{2h} H([v_n]_\beta, [v]_\beta)^p \, d\beta \right)^{1/p} + \left( \int_h^{2h} H([v]_\beta, [v]_{\beta-h})^p \, d\beta \right)^{1/p}$$

$$\leq \left( \int_h^{1} H([v_n]_{\beta-h}, [v_n]_\beta)^p \, d\beta \right)^{1/p} + \left( \int_h^{1} H([v_n]_\beta, [v]_\beta)^p \, d\beta \right)^{1/p} + \left( \int_h^{1} H([v]_\beta, [v]_{\beta-h})^p \, d\beta \right)^{1/p}. \tag{14}$$

Since $U$ is $p$-mean equi-left-continuous, there exists an $h \in (0, 1/2)$ such that

$$\left( \int_h^{1} H([v_n]_{\beta-h}, [v_n]_\beta)^p \, d\beta \right)^{1/p} < \varepsilon/4 \tag{15}$$

for all $n = 1, 2, \ldots$. From (8), we know if $n \to \infty$ then $H([v_n]_{\beta-h}, [v_n]_\beta) \to H([v]_{\beta-h}, [v]_\beta)$ a.e. on $\beta \in [h, 1]$. So, by Fatou Lemma, we have

$$\left( \int_h^{1} H([v]_{\beta-h}, [v]_\beta)^p \, d\beta \right)^{1/p} \leq \liminf_n \left( \int_h^{1} H([v_n]_{\beta-h}, [v_n]_\beta)^p \, d\beta \right)^{1/p} \leq \varepsilon/4, \tag{16}$$

Note that $[v_n]_h$ and $[v]_h$ are contained in $\{[u]_\alpha : u \in U\}$ which is compact, it thus follows from the Lebesgue's dominated convergence theorem and (8) that

$$\left( \int_h^{1} H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p} \to 0$$

as $n \to \infty$. Hence there is an $N(h, \varepsilon)$ such that

$$\left( \int_h^{1} H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p} \leq \varepsilon/4 \tag{17}$$

for all $n \geq N$.

Combined with (14), (15), (16), and (17), it yields that

$$\left( \int_0^{1} H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p}$$

$$\leq \left( \int_0^{h} H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p} + \left( \int_h^{1} H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p}$$

$$\leq \varepsilon/4 + \left( \int_h^{1} H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p} + \varepsilon/4 + \left( \int_h^{1} H([v_n]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p}$$

$\leq \varepsilon$

for all $n \geq N$. Thus we obtain (13) from the arbitrariness of $\varepsilon$.

**Step 3.** Show that $v \in F_B(\mathbb{R}^m)^p$.

By (13), we know that there is an $N$ such that

$$\left(\int_0^1 H([v]_\alpha, [v_N]_\alpha)^p \, d\alpha\right)^{1/p} < 1,$$

and then

$$\begin{aligned} &\left(\int_0^1 H([v]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} \\ &\leq \left(\int_0^1 H([v]_\alpha, [v_N]_\alpha)^p \, d\alpha\right)^{1/p} + \left(\int_0^1 H([v_N]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} \\ &\leq 1 + \left(\int_0^1 H([v_N]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty. \end{aligned}$$

By properties (i),(ii) and (iii) of $v_\alpha$ and Theorem 3.9, this yields that $v \in F_B(\mathbb{R}^m)^p$.

From steps 1, 2 and 3, we know that for arbitrary sequence $\{u_n\}$ of $U$, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ which converges to $v \in F_B(\mathbb{R}^m)^p$. This means that $U$ is a relatively compact set in $(F_B(\mathbb{R}^m)^p, d_p)$. □

By using the characterization of relatively compact sets in $(F_B(\mathbb{R}^m)^p, d_p)$ given in Theorem 4.1, we can derive characterizations of totally bounded sets and compact sets below as consequences.

**Theorem 4.2.** *$U$ is a totally bounded set in $(F_B(\mathbb{R}^m)^p, d_p)$ if and only if*
*(i) $U$ is uniformly $p$-mean bounded;*
*(ii) $U$ is $p$-mean equi-left-continuous.*

**Proof.** Notice that it only use the totally boundedness of $U$ to show the necessity part of the proof of Theorem 4.1. So the desired conclusion follows immediately from Theorem 4.1. □

**Theorem 4.3.** *Let $U$ be a subset of $(F_B(\mathbb{R}^m)^p, d_p)$, then $U$ is compact in $(F_B(\mathbb{R}^m)^p, d_p)$ if and only if*
*(i) $U$ is uniformly $p$-mean bounded;*
*(ii) $U$ is $p$-mean equi-left-continuous;*
*(iii) $U$ is a closed set in $(F_B(\mathbb{R}^m)^p, d_p)$.*

**Proof.** The desired result follows immediately from Theorem 4.1. □

## 5. Relationship between $(\boldsymbol{F_B}(\mathbb{R}^m), \boldsymbol{d_p})$ and $(\boldsymbol{F_B}(\mathbb{R}^m)^p, \boldsymbol{d_p})$ and properties of $(\boldsymbol{F_B}(\mathbb{R}^m), \boldsymbol{d_p})$

In this section, we show that $(F_B(\mathbb{R}^m)^p, d_p)$ is the completion of $(F_B(\mathbb{R}^m), d_p)$, and then present characterizations of totally bounded sets, relatively compact sets and compact sets in $(F_B(\mathbb{R}^m), d_p)$.

Diamond and Kloeden [9] pointed out that $(E^m, d_p)$ is not a complete space. Ma [25] gave the following example to show this fact. Let

$$u_n(x) = \begin{cases} e^{-x}, & \text{if } 0 \le x \le n, \\ 0, & \text{otherwise,} \end{cases} \quad n = 1, 2, \ldots.$$

Then $\{u_n\}$ is a Cauchy sequence in $(E^1, d_1) \cap (S_0^1, d_1)$. Put

$$u(x) = \begin{cases} e^{-x}, & \text{if } 0 \le x < +\infty, \\ 0, & \text{otherwise,} \end{cases}$$

then it can be checked that $u \in F_B(\mathbb{R}^1)^1 \backslash F_B(\mathbb{R}^1)$ and $u_n$ converges to $u$ in $d_1$ metric. Note that $E^1, S_0^1 \subset S^1 \subset \widetilde{S}^1 \subset F_B(\mathbb{R}^1)$, it yields that none of $(E^1, d_1)$, $(S_0^1, d_1)$, $(S^1, d_1)$, $(\widetilde{S}^1, d_1)$ and $(F_B(\mathbb{R}^1), d_1)$ is complete. Along this way, we can show that none of $(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$, $(\widetilde{S}^m, d_p)$ and $(F_B(\mathbb{R}^m), d_p)$ is complete.

First, we do step by step to show that the completion of $(F_B(\mathbb{R}^m), d_p)$ is exactly its $L_p$-extension $(F_B(\mathbb{R}^m)^p, d_p)$. Two facts are first proved which are exhibited in Theorems 5.1 and 5.2.

**Theorem 5.1.** *$(F_B(\mathbb{R}^m)^p, d_p)$ is a complete space.*

**Proof.** It suffices to prove that each Cauchy sequence has a limit in $(F_B(\mathbb{R}^m)^p, d_p)$. Let $\{u_n : n \in \mathbb{N}\}$ be a Cauchy sequence in $(F_B(\mathbb{R}^m)^p, d_p)$, we assert that $\{u_n : n \in \mathbb{N}\}$ is a relatively compact set in $(F_B(\mathbb{R}^m)^p, d_p)$.

To show this assertion, by Theorem 4.1, it is equivalent to prove that $\{u_n : n \in \mathbb{N}\}$ is a bounded set in $(F_B(\mathbb{R}^m)^p, d_p)$ and that $\{u_n : n \in \mathbb{N}\}$ is $p$-mean equi-left-continuous. The former follows immediately from the fact that $\{u_n : n \in \mathbb{N}\}$ is a Cauchy sequence.

To prove the latter, suppose $\varepsilon > 0$. Since $\{u_n : n \in \mathbb{N}\}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ satisfies that $d_p(u_n, u_m) \le \varepsilon/3$ for all $n, m \ge N$. By Lemma 4.3, $\{u_k : 1 \le k \le N\}$ is $p$-mean equi-left-continuous, hence we can find an $h > 0$ such that

$$\left( \int_h^1 H([u_k]_{\alpha-h}, [u_k]_\alpha)^p \, d\alpha \right)^{1/p} \le \varepsilon/3 \tag{18}$$

for all $1 \leq k \leq N$. If $k > N$, then

$$\begin{aligned}
&\left(\int_h^1 H([u_k]_{\alpha-h}, [u_k]_\alpha)^p \, d\alpha\right)^{1/p} \\
&\leq \left(\int_h^1 H([u_k]_{\alpha-h}, [u_N]_{\alpha-h})^p \, d\alpha\right)^{1/p} + \left(\int_h^1 H([u_N]_{\alpha-h}, [u_N]_\alpha)^p \, d\alpha\right)^{1/p} + \left(\int_h^1 H([u_N]_\alpha, [u_k]_\alpha)^p \, d\alpha\right)^{1/p} \\
&\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \qquad (19)
\end{aligned}$$

From the arbitrariness of $\varepsilon$ and ineqs. (18) and (19), we know that $\{u_n : n \in \mathbb{N}\}$ is $p$-mean equi-left-continuous.

Now, from the relatively compactness of $\{u_n : n \in \mathbb{N}\}$ in $(F_B(\mathbb{R}^m)^p, d_p)$, there exists a subsequence $\{u_{n_k} : k = 1, 2, \ldots\}$ of $\{u_n : n \in \mathbb{N}\}$ such that $\lim_{k\to\infty} u_{n_k} = u \in F_B(\mathbb{R}^m)^p$. Note that $\{u_n : n \in \mathbb{N}\}$ is a Cauchy sequence, we thus know that $u_n$, $n = 1, 2, \ldots$, also converges to $u$ in $(F_B(\mathbb{R}^m)^p, d_p)$.

The proof is completed. □

**Remark 5.1.** By Theorems 4.1 and 4.2, a set $U$ in $(F_B(\mathbb{R}^m)^p, d_p)$ is totally bounded if and only if it is relatively compact. This fact alone can ensure that $(F_B(\mathbb{R}^m)^p, d_p)$ is complete.

**Theorem 5.2.** *$F_B(\mathbb{R}^m)$ is a dense set in $(F_B(\mathbb{R}^m)^p, d_p)$.*

**Proof.** Given $u \in F_B(\mathbb{R}^m)^p$. Put $u_n = u^{(1/n)}$, $n = 1, 2, \ldots$. Then

$$[u_n]_\alpha = \begin{cases} [u]_\alpha, & \text{if } \alpha \geq 1/n, \\ [u]_{1/n}, & \text{if } \alpha \leq 1/n, \end{cases}$$

for all $\alpha \in [0, 1]$. Notice that $[u_n]_0 = [u]_{1/n} \in K(\mathbb{R}^m)$. It thus follows from Theorems 3.4 and 3.9 that $u_n \in F_B(\mathbb{R}^m)$ for $n = 1, 2, \ldots$.

Since $u \in F_B(\mathbb{R}^m)^p$, we know $\left(\int_0^1 H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < +\infty$, thus, by the absolute continuity of the Lebesgue's integral, it holds that, for each $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that

$$\left(\int_0^\delta H([u]_\alpha, \{0\})^p \, d\alpha\right)^{1/p} < \varepsilon. \qquad (20)$$

Note that

$$d_p(u_n, u) = \left(\int_0^{1/n} H([u]_{1/n}, [u]_\alpha)^p \, d\alpha\right)^{1/p}$$

$$
\begin{aligned}
&\leq \left( \int_0^{1/n} H([u]_\alpha, \{0\})^p \, d\alpha \right)^{1/p} + \left( \int_0^{1/n} H([u]_{1/n}, \{0\})^p \, d\alpha \right)^{1/p} \\
&\leq 2 \left( \int_0^{1/n} H([u]_\alpha, \{0\})^p \, d\alpha \right)^{1/p},
\end{aligned}
$$

it then follows from ineq.(20) that $d_p(u_n, u) \to 0$ as $n \to \infty$.

So, for each $u \in F_B(\mathbb{R}^m)^p$, we can find a sequence $\{u_n\} \subset F_B(\mathbb{R}^m)$ such that $u_n$ converges to $u$. This means that $F_B(\mathbb{R}^m)$ is dense in $F_B(\mathbb{R}^m)^p$. $\square$

**Remark 5.2.** From the proof of Theorem 5.2, we know the following fact. Given $u \in F_B(\mathbb{R}^m)^p$, then $u^{(1/n)} \in F_B(\mathbb{R}^m)$ for each $n \in \mathbb{N}$, and $d_p(u^{(1/n)}, u) \to 0$ as $n \to \infty$.

Now, we get relationship between $(F_B(\mathbb{R}^m)^p, d_p)$ and $(F_B(\mathbb{R}^m), d_p)$.

**Theorem 5.3.** *$(F_B(\mathbb{R}^m)^p, d_p)$ is the completion of $(F_B(\mathbb{R}^m), d_p)$.*

**Proof.** The desired result follows immediately from Theorems 5.1 and 5.2. $\square$

Next, we exhibit some characterizations of totally bounded sets, relatively compact sets and compact sets in $(F_B(\mathbb{R}^m), d_p)$ which are consequences of the conclusions in Section 4.

**Theorem 5.4.** *Let $U \subset F_B(\mathbb{R}^m)$, then $U$ is totally bounded in $(F_B(\mathbb{R}^m), d_p)$ if and only if*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous.*

**Proof.** Note that $F_B(\mathbb{R}^m) \subset F_B(\mathbb{R}^m)^p$, we thus know $U \subset F_B(\mathbb{R}^m)$ is a totally bounded set in $(F_B(\mathbb{R}^m), d_p)$ if and only if $U$ is a totally bounded set in $(F_B(\mathbb{R}^m)^p, d_p)$. So the desired conclusion follows immediately from Theorem 4.2. $\square$

Given $U \in F_B(\mathbb{R}^m)^p$, for writing convenience, we use $\overline{U}$ to denote the closure of $U$ in $(F_B(\mathbb{R}^m)^p, d_p)$.

**Theorem 5.5.** *Let $U \subset F_B(\mathbb{R}^m)$, then $U$ is relatively compact in $(F_B(\mathbb{R}^m), d_p)$ if and only if*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous;*
*(iii) $\overline{U} \subset F_B(\mathbb{R}^m)$.*

**Proof.** The desired conclusion follows immediately from Theorem 4.1 and the obvious fact that $F_B(\mathbb{R}^m) \subset F_B(\mathbb{R}^m)^p$. $\square$

**Theorem 5.6.** *Let $U \subset F_B(\mathbb{R}^m)$, then $U$ is compact in $(F_B(\mathbb{R}^m), d_p)$ if and only if*
*(i) $U$ is uniformly $p$-mean bounded;*
*(ii) $U$ is $p$-mean equi-left-continuous;*
*(iii) $\overline{U} = U$.*

**Proof.** The desired result follows immediately from Theorem 4.3. $\square$

Condition (iii) in Theorem 5.6 involves the closure of $U$ in the completion space $(F_B(\mathbb{R}^m)^p, d_p)$. We intend to find another characterization of compactness that depends only on $U$ itself, which is the last result of this section. To establish this new characterization of compactness, we need the following concept.

Let $B_r := \{x \in \mathbb{R}^m : \|x\| \leq r\}$, where $r$ is a positive real number. $\widehat{B_r}$ denotes the characteristic function of $B_r$. Given $u \in F_B(\mathbb{R}^m)$, then $u \vee \widehat{B_r} \in F_B(\mathbb{R}^m)$. Define

$$|u|^r := \left( \int_0^1 H([u \vee \widehat{B_r}]_\alpha, [\widehat{B_r}]_\alpha)^p \, d\alpha \right)^{1/p}.$$

It can be checked that, for $u \in F_B(\mathbb{R}^m)$, $|u|^r = 0$ if and only if $[u]_0 \subseteq B_r$. Note that

$$H([u \vee \widehat{B_r}]_\alpha, [v \vee \widehat{B_r}]_\alpha) \leq H([u]_\alpha, [v]_\alpha),$$

it thus holds that

$$d_p(u, v) \geq ||u|^r - |v|^r|. \tag{21}$$

**Theorem 5.7.** *Let $U \subset F_B(\mathbb{R}^m)$, then $U$ is relatively compact in $(F_B(\mathbb{R}^m), d_p)$ if and only if $U$ satisfies conditions (i), (ii) in Theorem 4.1 and the following condition (iii′).*

**(iii′)** *Given $\{u_n : n = 1, 2, \ldots\} \subset U$, there exists a $r > 0$ and a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\lim_{n\to\infty} |v_n|^r = 0$.*

**Proof.** Suppose that $U$ is a relatively compact set but does not satisfy condition (iii′). Take $r = 1$, then there exists $\varepsilon_1 > 0$ and a subsequence $\{u_n^{(1)} : n = 1, 2, \ldots\}$ of $\{u_n : n = 1, 2, \ldots\}$ such that $|u_n^{(1)}|^1 > \varepsilon_1$ for all

$n = 1, 2, \ldots$. If $\{u_n^{(1)}\}, \ldots, \{u_n^{(k)}\}$ and positive numbers $\varepsilon_1, \ldots, \varepsilon_k$ have been chosen, we can find a subsequence $\{u_n^{(k+1)}\}$ of $\{u_n^{(k)}\}$ and $\varepsilon_{k+1} > 0$ such that $|u_n^{(k+1)}|^{k+1} > \varepsilon_{k+1}$ for all $n = 1, 2, \ldots$. Put $v_n = u_n^{(n)}$ for $n = 1, 2, \ldots$. Then $\{v_n\}$ is a subsequence of $\{u_n\}$ and

$$\liminf_{n\to\infty} |v_n|^k \geq \varepsilon_k \tag{22}$$

for $k = 1, 2, \ldots$. Let $v \in F_B(\mathbb{R}^m)^p$ be a accumulation point of $\{v_n\}$. It then follows from (21) and (22) that

$$|v|^k \geq \varepsilon_k > 0$$

for all $k = 1, 2, \ldots$. So we know $v \notin F_B(\mathbb{R}^m)$. This contradicts the fact that $U$ is a relatively compact set in $(F_B(\mathbb{R}^m), d_p)$.

Suppose that $U \subset (F_B(\mathbb{R}^m), d_p)$ satisfies condition (iii′). Given a sequence $\{u_n\}$ in $U$ with $\lim_{n\to\infty} u_n = u \in F_B(\mathbb{R}^m)^p$. Then, from (21), there exists a $r > 0$ such that $\lim_{n\to\infty} |u_n|^r = |u|^r = 0$. Hence $[u]_0 \subseteq B_r$, i.e. $u \in F_B(\mathbb{R}^m)$. So, by Theorem 4.1, we know that if $U$ meets conditions (i), (ii) and (iii′), then $U$ is a relatively compact set in $(F_B(\mathbb{R}^m), d_p)$. □

**Theorem 5.8.** *Let $U$ be a set in $F_B(\mathbb{R}^m)$, then $U$ is compact in $(F_B(\mathbb{R}^m), d_p)$ if and only if $U$ is closed in $(F_B(\mathbb{R}^m), d_p)$ and satisfies conditions (i), (ii) and (iii′) in Theorem 5.7.*

**Proof.** The desired result follows immediately from Theorem 5.7. □

## 6. Subspaces of $(F_B(\mathbb{R}^m)^p, d_p)$

We have already shown that $(F_B(\mathbb{R}^m)^p, d_p)$ is the completion of $(F_B(\mathbb{R}^m), d_p)$. We might expect that $(E^{m,p}, d_p)$, $(S_0^{m,p}, d_p)$, $(S^{m,p}, d_p)$ and $(\widetilde{S}^{m,p}, d_p)$ are the completions of $(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$ and $(\widetilde{S}^m, d_p)$, respectively. In this section, we prove that this is true and discuss relationship among all these fuzzy sets spaces. The conclusions are summarized in a figure. Then, by using these conclusions and the results in Sections 4 and 5, we gives characterizations of totally boundedness, relatively compactness and compactness of all subspaces of $(F_B(\mathbb{R}^m)^p, d_p)$ mentioned in this paper.

First, we give a series of conclusions on relationships among subspaces of $(F_B(\mathbb{R}^m)^p, d_p)$, which will be summarized in a figure.

**Theorem 6.1.** *$\widetilde{S}^{m,p}$ is a closed set in $(F_B(\mathbb{R}^m)^p, d_p)$.*

**Proof.** It only need to show that each accumulation point of $\widetilde{S}^{m,p}$ belongs to itself. Given a sequence $\{u_n\}$ in $\widetilde{S}^{m,p}$ with $\lim u_n = u \in F_B(\mathbb{R}^m)^p$, then clearly $H([u_n]_\alpha, [u]_\alpha) \stackrel{\text{a.e.}}{\to} 0$ $([0,1])$. Suppose that $\alpha \in (0,1]$. If $H([u_n]_\alpha, [u]_\alpha) \to 0$, then by Theorem 2.1, $[u]_\alpha \in K_S(\mathbb{R}^m)$. If $H([u_n]_\alpha, [u]_\alpha) \not\to 0$, then there exists a sequence $\beta_n \to \alpha-$ such that $[u]_{\beta_n} \in K_S(\mathbb{R}^m)$. Note that $[u]_\alpha = \bigcap_n [u]_{\beta_n}$, this implies that $[u]_\alpha \in K_S(\mathbb{R}^m)$. So we have $u \in \widetilde{S}^{m,p}$. □

**Theorem 6.2.** *$\widetilde{S}^m$ is a dense set in $(\widetilde{S}^{m,p}, d_p)$.*

**Proof.** Given $u \in \widetilde{S}^{m,p}$. Put $u_n = u^{(1/n)}$, $n = 1, 2, \ldots$. Then $\{u_n\} \subset \widetilde{S}^m$. From Remark 5.2, we know that $d_p(u_n, u) \to 0$ as $n \to \infty$. So $\widetilde{S}^m$ is dense in $(\widetilde{S}^{m,p}, d_p)$. □

**Theorem 6.3.** *$S^{m,p}$ is a closed subset of $(\widetilde{S}^{m,p}, d_p)$.*

**Proof.** To show that $S^{m,p}$ is a closed set in $(\widetilde{S}^{m,p}, d_p)$, let $\{u_n\}$ be a sequence in $S^{m,p}$ which converges to $u \in \widetilde{S}^{m,p}$, we only need to prove that $u \in S^{m,p}$.

Since $d_p(u_n, u) = \left(\int_0^1 H([u_n]_\alpha, [u]_\alpha)^p \, d\alpha\right)^{1/p} \to 0$, it holds that

$$H([u_n]_\alpha, [u]_\alpha) \to 0 \text{ a.e. on } [0,1]. \tag{23}$$

Hence $\{\ker u_n : n = 1, 2, \ldots\}$ is a bounded set in $(K(\mathbb{R}^m), H)$, and therefore

$$\limsup_{n\to\infty} \ker u_n \neq \emptyset. \tag{24}$$

We assert that

$$\limsup_{n\to\infty} \ker u_n \subset \ker [u]_\alpha \text{ for all } \alpha \in (0,1]. \tag{25}$$

So, from (24) and (25), we know

$$\emptyset \neq \limsup_{n\to\infty} \ker u_n \subset \bigcap_{\alpha\in(0,1]} \ker [u]_\alpha = \ker u.$$

It thus follows from Theorem 3.7 that $u \in S^{m,p}$.

Now we prove (25). The proof is divided into two cases.
**Case 1**. $\alpha \in (0,1]$ satisfies the condition that $H([u_n]_\alpha, [u]_\alpha) \to 0$.
In this case, by Corollary 2.1, we have that

$$\limsup_{n\to\infty} \ker\, u_n \subset \limsup_{n\to\infty} \ker\, [u_n]_\alpha \subset \ker\, [u]_\alpha.$$

**Case 2**. $\alpha \in (0,1]$ satisfies the condition that $H([u_n]_\alpha, [u]_\alpha) \not\to 0$.
By (23), we know that there is a sequence $\alpha_n \in (0,1]$, $n = 1, 2, \ldots$, such that $\alpha_n \to \alpha-$ and $H([u_n]_{\alpha_n}, [u]_{\alpha_n}) \to 0$. From case 1, we obtain that

$$\limsup_{n\to\infty} \ker\, u_n \subset \bigcap_{n=1}^{+\infty} \ker\, [u]_{\alpha_n}. \tag{26}$$

Note that $H([u]_\alpha, [u]_{\alpha_n}) \to 0$, so, by Corollary 2.1,

$$\limsup_{n\to\infty} \ker\, [u]_{\alpha_n} \subset \ker\, [u]_\alpha, \tag{27}$$

combined (26) and (27), we get that

$$\limsup_{n\to\infty} \ker\, u_n \subset \ker\, [u]_\alpha.$$

□

**Corollary 6.1.** *Suppose that $\{u_n\}$ is a sequence in $(S^{m,p}, d_p)$ and that $u \in \widetilde{S}^{m,p}$. If $d_p(u_n, u) \to 0$, then $u \in S^{m,p}$ and $\limsup_{n\to\infty} ker\, u_n \subset ker\, u$.*

**Proof.** The desired result follows immediately from the proof of Theorem 6.3. □

**Corollary 6.2.** *$(S^m, d_p)$ is a closed subspace of $(\widetilde{S}^m, d_p)$.*

**Proof.** The desired result follows immediately from Corollary 6.1. □

**Theorem 6.4.** *$S^m$ is a dense set in $(S^{m,p}, d_p)$.*

**Proof.** The proof is similar to the proof of Theorem 5.2. □

**Theorem 6.5.** *$(E^{m,p}, d_p)$ is a closed subspace of $(S^{m,p}, d_p)$.*

**Proof.** By Proposition 2.1, we know that $(K_C(\mathbb{R}^m), H)$ is a closed set in $(K_S(\mathbb{R}^m), H)$. In a way similar to the proof of Theorem 6.3, we can obtain the desired result by using this fact. □

$$
\begin{tikzcd}
(S_0^m, d_p), \arrow[d] & (E^m, d_p) \arrow[r, phantom, "\prec"] \arrow[d] & (S^m, d_p) \arrow[r, phantom, "\prec"] \arrow[d] & (\tilde{S}^m, d_p) \arrow[r, phantom, "\prec"] \arrow[d] & (F_B(R^m), d_p) \arrow[d] \\
(S_0^{m,p}, d_p), & (E^{m,p}, d_p) \arrow[r, phantom, "\prec"] & (S^{m,p}, d_p) \arrow[r, phantom, "\prec"] & (\tilde{S}^{m,p}, d_p) \arrow[r, phantom, "\prec"] & (F_B(R^m)^p, d_p)
\end{tikzcd}
$$

Figure 1: Relationship among various subspaces of $(F(\mathbb{R}^m)^p, d_p)$.
$A \prec B$ denotes that $A$ is a closed subspace of $B$ and $A \longrightarrow B$ means that $B$ is the completion of $A$.

**Corollary 6.3.** *$(E^m, d_p)$ is a closed subspace of $(S^m, d_p)$.*

**Proof.** The desired results follows immediately from Theorem 6.5. □

**Theorem 6.6.** *$E^m$ is a dense set in $(E^{m,p}, d_p)$.*

**Proof.** The proof is similar to the proof of Theorem 6.4. □

**Theorem 6.7.** *$(\widetilde{S}^{m,p}, d_p)$ is the completion of $(\widetilde{S}^m, d_p)$.*

**Proof.** The desired result follows from Theorems 5.1, 6.1, 6.2. □

**Theorem 6.8.** *$(S^{m,p}, d_p)$ is the completion of $(S^m, d_p)$.*

**Proof.** The desired result follows from Theorems 6.3, 6.4, 6.7. □

**Theorem 6.9.** *$(E^{m,p}, d_p)$ is the completion of $(E^m, d_p)$.*

**Proof.** The desired result follows from Theorems 6.5, 6.6, 6.8. □

From Corollary 6.1 and Remark 5.2, we can obtain the following two theorems.

**Theorem 6.10.** *$(S_0^{m,p}, d_p)$ is the completion of $(S_0^m, d_p)$.*

**Theorem 6.11.** *$(S_0^m, d_p)$ is a closed subspace of $(S^m, d_p)$.*

Figure 1 summarizes all the above results in this section.

Krätschmer [32] presented the completion of $(E^m, d_p)$ which is generated and described via the support functions of fuzzy numbers. In this paper, we find that the completions of $(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$, $(\widetilde{S}^m, d_p)$ and $(F_B(\mathbb{R}^m), d_p)$ can be obtained by means of $L_p$-extension. To a certain extent,

completions illustrated in this form are more concise and clear and so more easy to perform theoretical research and practical application.

Now, based on these statements about relationship among subspaces of $(F(\mathbb{R}^m)^p, d_p)$, and characterizations of compactness given in Sections 4 and 5, we intend to give characterizations of totally bounded sets, relatively compact sets and compact sets in subspaces of $(F(\mathbb{R}^m)^p, d_p)$.

**Theorem 6.12.** *Let $U \subset \widetilde{S}^{m,p}$ ($U \subset E^{m,p}$, $U \subset S_0^{m,p}$, $U \subset S^{m,p}$), then $U$ is totally bounded if and only if it is relatively compact in $(\widetilde{S}^{m,p}, d_p)$ ($(E^{m,p}, d_p)$, $(S_0^{m,p}, d_p)$, $(S^{m,p}, d_p)$), which is equivalent to*
*(i) $U$ is uniformly $p$-mean bounded, and*
*(ii) $U$ is $p$-mean equi-left-continuous.*

**Proof.** Note that in a complete space, a set is totally bounded if and only if it is relatively compact. So the desired results follow from Theorems 4.1 and the completeness of $(\widetilde{S}^{m,p}, d_p)$, $(E^{m,p}, d_p)$, $(S_0^{m,p}, d_p)$ and $(S^{m,p}, d_p)$. $\square$

Let $U \subset \widetilde{S}^{m,p}$ ($U \subset E^{m,p}$, $U \subset S_0^{m,p}$, $U \subset S^{m,p}$). It is easy to see that $\overline{U}$ is exactly the closure of $U$ in $\widetilde{S}^{m,p}$ ($E^{m,p}$, $S_0^{m,p}$, $S^{m,p}$).

**Theorem 6.13.** *Let $U$ be a set in $(\widetilde{S}^{m,p}, d_p)$ ($(E^{m,p}, d_p)$, $(S_0^{m,p}, d_p)$, $(S^{m,p}, d_p)$), then $U$ is compact in $(\widetilde{S}^{m,p}, d_p)$ ($(E^{m,p}, d_p)$, $(S_0^{m,p}, d_p)$, $(S^{m,p}, d_p)$) if and only if*
*(i) $U$ is uniformly $p$-mean bounded;*
*(ii) $U$ is $p$-mean equi-left-continuous;*
*(iii) $U = \overline{U}$.*

**Proof.** The desired results follow immediately from Theorem 6.12. $\square$

**Theorem 6.14.** *Let $U \subset \widetilde{S}^m$ ($U \subset E^m$, $U \subset S_0^m$, $U \subset S^m$), then $U$ is totally bounded in $(\widetilde{S}^m, d_p)$ ($(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$) if and only if*
*(i) $U$ is uniformly $p$-mean bounded;*
*(ii) $U$ is $p$-mean equi-left-continuous.*

**Proof.** The desired conclusion follows immediately from Theorem 4.2. $\square$

**Theorem 6.15.** *Let $U \subset \widetilde{S}^m$ ($U \subset E^m$, $U \subset S_0^m$, $U \subset S^m$), then $U$ is relatively compact in $(\widetilde{S}^m, d_p)$ ($(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$) if and only if*
*(i) $U$ is uniformly $p$-mean bounded;*
*(ii) $U$ is $p$-mean equi-left-continuous;*
*(iii) $\overline{U} \subset \widetilde{S}^m$ ($\overline{U} \subset E^m$, $\overline{U} \subset S_0^m$, $\overline{U} \subset S^m$).*

**Proof.** The desired conclusion follows immediately from Theorem 6.12. $\square$

**Theorem 6.16.** *Let $U \subset \widetilde{S}^m$ ($U \subset E^m$, $U \subset S_0^m$, $U \subset S^m$), then $U$ is compact in $(\widetilde{S}^m, d_p)$ ($(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$) if and only if*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous;*
*(iii) $\overline{U} = U$.*

**Proof.** The desired results follow from Theorem 4.3 and the completeness of $(\widetilde{S}^{m,p}, d_p)$, $(E^{m,p}, d_p)$, $(S_0^{m,p}, d_p)$ and $(S^{m,p}, d_p)$. $\square$

**Theorem 6.17.** *Let $U \subset \widetilde{S}^m$ ($U \subset E^m$, $U \subset S_0^m$, $U \subset S^m$), then $U$ is relatively compact in $(\widetilde{S}^m, d_p)$ ($(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$) if and only if*
*(i) $U$ is uniformly p-mean bounded;*
*(ii) $U$ is p-mean equi-left-continuous;*
*(iii′) Given $\{u_n : n = 1, 2, \ldots\} \subset U$, there exists a $r > 0$ and a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\lim_{n\to\infty} |v_n|^r = 0$.*

**Proof.** Note that $F_B(\mathbb{R}^m) \cap \widetilde{S}^{m,p} = \widetilde{S}^m$, $F_B(\mathbb{R}^m) \cap E^{m,p} = E^m$, $F_B(\mathbb{R}^m) \cap S_0^{m,p} = S_0^m$, $F_B(\mathbb{R}^m) \cap S^{m,p} = S^m$, so we can obtain the desired results by applying Theorems 5.7 and 6.12. $\square$

**Theorem 6.18.** *Let $U$ be a set in $\widetilde{S}^m$ ($E^m$, $S_0^m$, $S^m$), then $U$ is compact in $(\widetilde{S}^m, d_p)$ ($(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$) if and only if $U$ is closed in $(\widetilde{S}^m, d_p)$ ($(E^m, d_p)$, $(S_0^m, d_p)$, $(S^m, d_p)$) and satisfies conditions (i), (ii) and (iii′) in Theorem 6.17.*

**Proof.** The desired result follows immediately from Theorem 6.17. $\square$

## 7. Applications of the results in this paper

In this section, by using results in this paper, we consider some properties of $d_p$ convergence on fuzzy sets spaces. Then we relook the characterizations of compactness presented in [35, 37].

Now we discuss properties of $d_p$ convergence from the point of view of level sets of fuzzy sets.

Let $\{u_k,\ k = 1, 2, \ldots\} \subset F_{GB}(\mathbb{R}^m)$ and $u \in F_{GB}(\mathbb{R}^m)$. For writing convenience, the symbol $H([u_k]_\alpha, [u]_\alpha) \xrightarrow{\text{a.e.}} 0\ (P)$ is used to denote that $H([u_k]_\alpha, [u]_\alpha) \to 0$ as $k \to \infty$ holds almost everywhere on $\alpha \in P$.

**Lemma 7.1.** *Suppose that* $\{u_k,\ k=1,2,\ldots\} \subset F_{GB}(\mathbb{R}^m)$, $u \in F_{GB}(\mathbb{R}^m)$ *and* $\alpha \in (0,1]$. *Then* $d_p(u_k^{(\alpha)}, u) \to 0$ *is equivalent to*
*(i)* $H([u_k]_\beta, [u]_\beta) \overset{a.e.}{\to} 0 \ ([\alpha, 1])$.
*(ii)* $H([u_k]_\alpha, [u]_\alpha) \to 0$ *and* $[u]_\gamma \equiv [u]_\alpha$ *when* $\gamma \in [0, \alpha]$.

**Proof.** The desired conclusion follows from the definition of $d_p$ metric and $u_k^{(\alpha)}$. □

**Remark 7.1.** Take $\alpha \in (0,1]$. From Lemma 7.1, we know that even if $\{u_k\}$ is a convergent sequence in $(F_B(\mathbb{R}^m)^p, d_p)$, $\{u_k^{(\alpha)}\}$ is not necessarily a convergent sequence in $(F_B(\mathbb{R}^m)^p, d_p)$. The following example is given to show this fact. This example is a small change of Example 4.1 in [35].

**Example 7.1.** Consider a sequence $\{u_n : n = 1, 2, \ldots\}$ in $E^1$ defined by

$$u_n(x) = \begin{cases} 1-2x, & x \in [0, \frac{1}{3}], \\ \frac{1}{3} - \frac{3}{2(n+3)}\left(x - \frac{1}{3}\right), & x \in [\frac{1}{3}, 1], \\ 0, & x \notin [0,1], \end{cases} \quad n = 1, 3, 5, \ldots,$$

$$u_n(x) = \begin{cases} 1-2x, & x \in [0, \frac{1}{3}], \\ \frac{1}{3}, & x \in [\frac{1}{3}, \frac{2}{3}], \\ \frac{1}{3} - \frac{3}{(n+3)}\left(x - \frac{2}{3}\right), & x \in [\frac{2}{3}, 1], \\ 0, & x \notin [0,1], \end{cases} \quad n = 2, 4, 6, \ldots.$$

It can be checked that $d_p(u_n, u_0) \to 0$, where

$$u_0(x) = \begin{cases} 1-2x, & x \in [0, \frac{1}{3}], \\ \frac{1}{3}, & x \in [\frac{1}{3}, 1], \\ 0, & x \notin [0,1]. \end{cases}$$

So we know that $\{u_n\}$ is a convergent sequence in $(E^1, d_p)$. But $\{u_n^{(\frac{1}{3})}\}$ is a divergent sequence in $(E^1, d_p)$. In fact, we can see that $d_p(u_{2n-1}^{(\frac{1}{3})}, v) \to 0$ and $d_p(u_{2n}^{(\frac{1}{3})}, w) \to 0$, where

$$v(x) = \begin{cases} 1-2x, & x \in [0, \frac{1}{3}], \\ 0, & x \notin [0, \frac{1}{3}], \end{cases}$$

$$w(x) = \begin{cases} 1-2x, & x \in [0, \frac{1}{3}], \\ \frac{1}{3}, & x \in [\frac{1}{3}, \frac{2}{3}], \\ 0, & x \notin [0,1]. \end{cases}$$

Clearly, $v \neq w$. Thus $\{u_n^{(\frac{1}{3})}\}$ is not a convergent sequence in $(E^1, d_p)$.

**Theorem 7.1.** *Suppose that $U \subset F_{GB}(\mathbb{R}^m)$ satisfying that $\{[u]_\alpha : u \in U\}$ is a bounded set in $K(\mathbb{R}^m)$ for each $\alpha \in (0,1]$. Let $\{r_i, i = 1,2,\ldots\}$ be a sequence in $(0,1]$. Then given $\{u_n\} \subset U$, it has a subsequence $\{u_{n_k}\}$ satisfying the following two statements.*
*(i) There exists a $u_0 \in F_{GB}(\mathbb{R}^m)$ such that $H([u_{n_k}]_\alpha, [u_0]_\alpha) \stackrel{a.e.}{\to} 0$ $([0,1])$.*
*(ii) There exist $u_{r_i}, i = 1,2,\ldots,$ in $K(\mathbb{R}^m)$ such that $H([u_{n_k}]_{r_i}, u_{r_i}) \to 0$ for each $r_i$.*

*Thus $d_p(u_{n_k}^{(r_i)}, u(r_i)) \to 0$ for each $r_i$, where $u(r_i) \in F_B(\mathbb{R}^m)$, $i = 1,2,\ldots,$ is defined by*

$$[u(r_i)]_\alpha = \begin{cases} [u_0]_\alpha, & \alpha \in (r_i, 1], \\ u_{r_i}, & \alpha \in [0, r_i]. \end{cases}$$

**Proof.** Given $\{u_n\} \subset U$. Since that $\{[u]_\alpha : u \in U\}$ is a bounded set in $K(\mathbb{R}^m)$ for each $\alpha \in (0,1]$, then from the proof of Theorem 4.1, we know that $\{u_n\}$ has a subsequence $\{u_{n_j}\}$ such that $H([u_{n_j}]_\alpha, [u_0]_\alpha) \stackrel{\text{a.e.}}{\to} 0$ $([0,1])$, where $u_0$ is one of the elements in $F_{GB}(\mathbb{R}^m)$.

Note that for each $\alpha > 0$, $\{[u]_\alpha : u \in U\}$ is a relatively compact set in $(K(\mathbb{R}^m), H)$. Now, by using the diagonal method and proceed according to the proof of Theorem 4.1, we can choose a subsequence $\{u_{n_k}\}$ of $\{u_{n_j}\}$ such that $H([u_{n_k}]_{r_i}, u_{r_i}) \to 0$ for each $r_i$, where $\{u_{r_i}, i = 1,2,\ldots\}$ is a sequence of elements in $K(\mathbb{R}^m)$.

So $\{u_{n_k}\}$ is a subsequence of $\{u_n\}$which satisfies statements (i) and (ii). Thus, by Lemma 7.1, we know that $d_p(u_{n_k}^{(r_i)}, u(r_i)) \to 0$ for all $r_i$. $\square$

**Remark 7.2.** Suppose that $U \subset F_B(\mathbb{R}^m)^p$ satisfies conditions (i) in Theorem 4.1, i.e., $U$ is uniformly $p$-mean bounded. Let $\{r_i, i = 1,2,\ldots\}$ be a sequence in $(0,1]$. Then given $\{u_n\} \subset U$, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}^{(r_i)}\}$ converges to $u(r_i) \in F_B(\mathbb{R}^m)$ in $d_p$ metric for each $r_i$.

In fact, if $U \subset F_B(\mathbb{R}^m)^p$ satisfies conditions (i) in Theorem 4.1, then $\{[u]_\alpha : u \in U\}$ is a bounded set in $K(\mathbb{R}^m)$ for each $\alpha \in (0,1]$. So above conclusion follows immediately from Theorem 7.1.

From Theorem 4.1 and Lemma 7.1, we can obtain two corollaries.

**Corollary 7.1.** *Suppose that $\{u_k,\ k = 1,2,\ldots\} \subset F_B(\mathbb{R}^m)^p$ and $u \in F_{GB}(\mathbb{R}^m)$. Then statements (i) and (ii) listed below are equivalent.*
*(i) There exists a decreasing sequence $\{r_i, i = 1,2,\ldots\}$ in $(0,1]$ converging to zero such that $d_p(u_k^{(r_i)}, u^{(r_i)}) \to 0$ for all $r_i$.*
*(ii) $H([u_k]_\alpha, [u]_\alpha) \stackrel{a.e.}{\to} 0$ $([0,1])$.*

*Furthermore, if $U := \{u_k,\ k = 1, 2, \ldots\}$ satisfies conditions (i) and (ii) in Theorem 4.1, then above statements (i) and (ii) are equivalent to statement (iii) below.*
*(iii)* $d_p(u_k, u) \to 0$ *and* $u \in F_B(\mathbb{R}^m)^p$.

**Corollary 7.2.** *Suppose that* $\{u_k,\ k = 1, 2, \ldots\} \subset F_B(\mathbb{R}^m)^p$, *that* $\{r_i, i = 1, 2, \ldots\}$ *is a decreasing sequence in* $(0, 1]$ *which converges to zero, and that* $u(r_i) \in F_{GB}(\mathbb{R}^m)$, $i = 1, 2, \ldots$. *Then the following statements are equivalent.*
*(i)* $d_p(u_k^{(r_i)}, u(r_i)) \to 0$ *for all* $r_i$.
*(ii) There exists a unique* $u_0 \in F_{GB}$ *such that* $H([u_k]_\alpha, [u_0]_\alpha) \xrightarrow{a.e.} 0$ $([0, 1])$ *and* $[u_0]_\alpha = [u_0^{(r_i)}]_\alpha = [u(r_i)]_\alpha$ *when* $r_i < \alpha \le 1$. *Moreover, for each* $r_i$, $H([u_k]_{r_i}, [u(r_i)]_{r_i}) \to 0$ *and* $[u(r_i)]_{r_i} \equiv [u(r_i)]_\theta$ *when* $\theta \in [0, r_i]$.

*Furthermore, if $U := \{u_k,\ k = 1, 2, \ldots\}$ satisfies conditions (i) and (ii) in Theorem 4.1, then above statements (i) and (ii) are equivalent to below statement (iii).*
*(iii) There exists a unique* $u_0 \in F_B(\mathbb{R}^m)^p$ *such that* $d_p(u_k, u_0) \to 0$ *and* $[u_0]_\alpha = [u_0^{(r_i)}]_\alpha = [u(r_i)]_\alpha$ *when* $r_i < \alpha \le 1$. *Moreover, for each* $r_i$, $H([u_k]_{r_i}, [u(r_i)]_{r_i}) \to 0$ *and* $[u(r_i)]_{r_i} \equiv [u(r_i)]_\theta$ *when* $\theta \in [0, r_i]$.

**Remark 7.3.** The readers can see [16, 31] for more studies on this topic, which consider the relations between $d_p$ convergence and other types of convergence on fuzzy sets spaces.

The foregoing results enable us to relook the characterizations of compactness proposed in previous work.

Compare Theorem 6.13 with Proposition 4.4. We can see that, in contrast to the former, the latter has an additional condition (iii):

> Let $r_i$ be a decreasing sequence in $(0, 1]$ converging to zero. For $\{u_k\} \subset U$, if $\{u_k^{(r_i)} : k = 1, 2, \ldots\}$ converges to $u(r_i) \in E^m$ in $d_p$ metric, then there exists a $u_0 \in E^m$ such that $[u_0^{(r_i)}]_\alpha = [u(r_i)]_\alpha$ when $r_i < \alpha \le 1$.

The reason is that $(E^m, d_p)$ is not complete. The function of "conditions (iii)" in Proposition 4.4 is to guarantee completeness of the closed subspace $(U, d_p)$ of $(E^m, d_p)$.

In fact, if $U$ satisfies conditions (i) and (ii) of Proposition 4.4, then from Remark 7.2, given $\{u_k\} \subseteq U$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_k\}$ such that $\{u_{n_k}^{(r_i)} : k = 1, 2, \ldots\}$ converges to $u(r_i) \in E^m$ in $d_p$ metric for each $r_i$.

Thus from Corollary 7.2, $\{u_{n_k}\}$ is a convergent sequence in $(E^{m,p}, d_p)$, and its limit point $u_0 \in E^{m,p}$ is defined by

$$[u_0]_\alpha = [u_0^{(r_i)}]_\alpha := [u(r_i)]_\alpha \text{ when } r_i < \alpha \leq 1,$$

for each $\alpha \in (0, 1]$. So it is easy to see that if $U$ satisfies conditions (i) and (ii) of Proposition 4.4, then condition (iii) in Proposition 4.4 is equivalent to $\overline{U} \subset E^m$.

Thus we know that Proposition 4.4 can be written in the following form:
**Proposition 4.4′** A closed set $U \subset (E^m, d_p)$ is compact if and only if:
(i) $U$ is uniformly $p$-mean bounded;
(ii) $U$ is $p$-mean equi-left-continuous;
(iii) $\overline{U} \subset E^m$.

Similarly, Propositions 4.5 and 4.6 can be written as:
**Proposition 4.5′** A closed set $U$ in $(S_0^m, d_p)$ is compact if and only if:
(i) $U$ is uniformly $p$-mean bounded;
(ii) $U$ is $p$-mean equi-left-continuous;
(iii) $\overline{U} \subset S_0^m$.
**Proposition 4.6′** A closed set $U$ in $(S^m, d_p)$ is compact if and only if:
(i) $U$ is uniformly $p$-mean bounded;
(ii) $U$ is $p$-mean equi-left-continuous;
(iii) $\overline{U} \subset S^m$.

## 8. Conclusions

We give a characterization of totally boundedness for fuzzy sets space endowed with $d_p$ metric. This is the key result of this paper. To prove this conclusion, we introduce the $L_p$-extension of a fuzzy sets space. Then we present a characterization of relatively compactness in $(F_B(\mathbb{R}^m)^p, d_p)$ which is the $L_p$-extension of $(F_B(\mathbb{R}^m), d_p)$. It's shown that this characterization is also a characterization of totally boundedness in $(F_B(\mathbb{R}^m)^p, d_p)$. From this fact, we know that this characterization is a characterization of totally boundedness in any subspaces of $(F_B(\mathbb{R}^m)^p, d_p)$ because totally boundedness is dependent only on the set itself and the metric $d_p$. From this fact, we can also deduce that $(F_B(\mathbb{R}^m)^p, d_p)$ is the completion of $(F_B(\mathbb{R}^m), d_p)$. Based on this, we give characterizations of relatively compactness and compactness in $(F_B(\mathbb{R}^m), d_p)$, $(F_B(\mathbb{R}^m)^p, d_p)$ and their subspaces. It is found that the completion of each fuzzy sets space mentioned in this paper is precisely its

$L_p$-extensions. Relationship among all the spaces mentioned in this paper are clarified and summarized in a figure. At last, as applications of results in this paper, we discuss some properties of $d_p$ convergence and relook characterizations of compactness proposed in previous work.

Since the input-output relation of fuzzy systems can be described using fuzzy functions [20, 33, 36], the results in this paper can be used to analysis and design fuzzy systems. Compactness criteria is used to solve fuzzy differential equations [29], so this topic is a potential application of our results.

## References


[1] R. A. Aliev, W. Pedrycz, B.G. Guirimov, R.R. Aliev, U. Ilhan, M. Babagil, S. Mammadli, Type-2 fuzzy neural networks with fuzzy clustering and differential evolution optimization, Information Sciences 181 (2011) 1591-1608.

[2] L.C. Barros, L.T. Gomes, P.A. Tonelli, Fuzzy differential equations: An approach via fuzzification of the derivative operator, Fuzzy Sets and Systems 230 (2013) 39-52.

[3] D.S. Bridges, Foundations of Real and Abstract Analysis, Springer, New York, 1998.

[4] J.J. Buckley, Y. Hayashi, Can fuzzy neural nets approximate continuous fuzzy functions, Fuzzy Sets and Systems 61 (1994) 43-51.

[5] J. Chanussot, I. Nyström, N. Sladoje, Shape signatures of fuzzy starshaped sets based on distance from the centroid, Pattern Recognition Letters 26 (2005) 735-746.

[6] S. Chen, J. Li, X. Wang, Fuzzy Set Theory and Its Application (in Chinese), Science Press, Beijing, 2005.

[7] P. Diamond, A note on fuzzy starshaped sets, Fuzzy Sets and Systems 37 (1990) 193-199.

[8] P. Diamond, P. Kloeden, Metric spaces of fuzzy sets, Fuzzy Sets and Systems 35 (1990) 241-249.

[9] P. Diamond, P. Kloeden, Metric Space of Fuzzy Sets–Theory and Application, World Scientific, Singapore, 1994.

[10] T. Fan, On the compactness of fuzzy numbers with sendograph, Fuzzy Sets and Systems 143 (2004) 471-477.

[11] G.H. Greco, M.P. Moschen, E. Quelho Frota Rezende, On the variational convergence for fuzzy sets in metric spaces, Ann. Univ. Ferrara Sez. VII Sc. Mat. 44 (1998) 27-39.

[12] G. H. Greco, Sendograph metric and relatively compact sets of fuzzy sets, Fuzzy Sets and Systems 157 (2006) 286-291.

[13] G.H. Greco, M.P. Moschen, Supremum metric and relatively compact sets of fuzzy sets, Nonlinear Analysis 64 (2006) 1325-1335.

[14] C. He, Y. Ye, Evolution cmputation based learning algorithms of polygonal fuzzy neural networks, International Journal of Intelligent Systems 26 (2011) 340-352.

[15] G. Huang, G.-B. Huang, S. Song, K. You, Trends in extreme learning machines: a review, Neural Networks 61 (2015) 32-48.

[16] H. Huang, C. Wu, Characterization of Γ-convergence on fuzzy number space, in: 11th International Fuzzy Systems Association World Congress, July 28-31, Beijing, China, Vol.1, 2005, pp. 66-70.

[17] H. Huang, C. Wu, Approximation capabilities of multilayer fuzzy neural networks on the set of fuzzy-valued functions, Information Sciences 179 (2009) 2762-2773.

[18] H. Huang, C. Wu, Approximation of fuzzy functions by regular fuzzy neural networks, Fuzzy Sets and Systems 177 (2011) 60-79.

[19] H. Huang, C. Wu, A new characterization of compact sets in fuzzy number spaces, Abstract and Applied Analysis, Volume 2013, http://dx.doi.org/10.1155/2013/820141.

[20] H. Huang, C. Wu, Approximation of fuzzy-valued functions by regular fuzzy neural networks and the accuracy analysis, Soft Computing 18 (2014) 2525-2540.

[21] H. Huang, Characterizations of endograph metric and Γ-convergence on fuzzy sets spaces, submitted.

[22] J. L. Kelley, General Topology, Springer, New York, NY, USA, 1975.

[23] E.P. Klement, M.L. Puri, D.A. Ralescu, Limit theorems for fuzzy random variables, Proceedings of the Royal Society of London, Series A 407 (1986) 171-182.

[24] P.E. Kloeden, T. Lorenz, A Peano theorem for fuzzy differential equations with evolving membership grade, Fuzzy Sets and Systems 280 (2015) 1-26.

[25] M. Ma, Some notes on the characterization of compact sets in $(E^n, d_p)$, Fuzzy Sets and Systems 56 (1993) 297-301.

[26] C.V. Negoita, D.A. Ralescu, Application of Fuzzy Sets to System Analysis, Wiley, New York, 1975.

[27] D. Qiu, L. Shu, Z.-W. Mo, On starshaped fuzzy sets, Fuzzy Sets and Systems 160 (2009) 1563-1577.

[28] H. Román-Flores, The compactness of $E(X)$, Applied Mathematics Letters 11 (1998) 13-17.

[29] H. Román-Flores, M. Rojas-Medar, Embedding of level-continuous fuzzy sets on Banach spaces, Information Sciences 144 (2002) 227-247.

[30] F. Scarselli, A.C. Tsoi, Universal approximation using feedforward neural networks: a survery of some existing methods, and some new results, Neural Networks 11 (1) (1998) 15-37.

[31] W. Trutschnig, Characterization of the sendograph-convergence of fuzzy sets by means of their $L_p$- and lelewise convergence, Fuzzy Sets and Systems 161 (2010) 1064-1077.

[32] V. Krätschmer, Some complete metrics on spaces of fuzzy subsets, Fuzzy Sets and Systems 130 (2002) 357-365.

[33] L. Wang, J. Mendel, Fuzzy basis functions, universal approximation, and orthogonal least-squares learning, IEEE Transactions on Neural Networks 3 (5) (1992) 807-814.

[34] C. Wu, M. Ma, The Basic of Fuzzy Analysis (in Chinese), National Defence Industry press, Beijing, 1991.

[35] C. Wu, Z. Zhao, Some notes on the characterization of compact sets of fuzzy sets with $L_p$ metric, Fuzzy Sets and Systems 159 (2008) 2104-2115.

[36] X. Zeng, M. Singh, Approximation accuracy analysis as function approxiamtors, IEEE Transactions on Fuzzy Systems 4 (1) (1996) 44-63.

[37] Z. Zhao, C. Wu, A characterization for compact sets in the space of fuzzy star-shaped numbers with $L_p$ metric, Abstract and Applied Analysis, Volume 2013, http://dx.doi.org/10.1155/2013/627314.